MATH 270
SUMMER 2004
FINAL EXAM

1. Consider the statement "If G is an abelian group, then every subgroup of G is an abelian group."

   a) (5 pt) Carefully state the contrapositive of this statement.
   b) (5 pt) Carefully state the converse of this statement.
   c) (5 pt) Prove the statement or give a counterexample.

2. Let (X, d) be a metric space.

   a) (5 pt) If $U_1, U_2, \cdots, U_n$ are open sets then show that $\bigcap_{k=1}^{n} U_k$ is open.
   b) (5 pt) If $C_1, C_2, \cdots, C_m$ are closed sets then $\bigcup_{k=1}^{m} C_k$ is closed.
   c) (5 pt) Do either of the above extend to the infinite case?

3. Consider the sequence

$$\{a_n\} = \left\{ \sqrt{3}, \sqrt{3+\sqrt{3}}, \sqrt{3+\sqrt{3+\sqrt{3}}}, \cdots \right\}.$$

   Inductively prove the following.

   a) (5 pt) $\{a_n\}$ is bounded.
   b) (5 pt) $\{a_n\}$ is increasing.

4. Let $\{A_i\}_{i\in\Lambda}$ be a family of sets.

   a) (5 pt) Show that $(\bigcup_{i\in\Lambda} A_i)^{\mathrm{c}} = \bigcap_{i\in\Lambda} A_i^{\mathrm{c}}$.
   b) (5 pt) Show that $(\bigcap_{i\in\Lambda} A_i)^{\mathrm{c}} = \bigcup_{i\in\Lambda} A_i^{\mathrm{c}}$ (you can use the previous result if you like).

5. Let $\mathfrak{C}$ denote the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. For two functions $f, g \in \mathfrak{C}$ we will say that $f \sim g$ if and only if $f - g$ is continuous.

   a) (5 pt) Show that $\sim$ defines an equivalence relation on $\mathfrak{C}$ (you may use the fact that sums and scalar multiples of continuous functions are continuous).
   b) (5 pt) Show that $\mathfrak{C}/\sim$ is not countable (hint: give an uncountable family of functions that are pairwise not equivalent).