MATH 270
SUMMER 2007
FINAL EXAM

1. Consider the statement $A \Longrightarrow B$.
   a) (5 pt) What is the converse of this statement?
   b) (5 pt) What is the contrapositive of this statement?

2. Let $n, m \in \mathbb{N}$.
   a) (5 pt) Prove that $2n \geq n + 1$ for all $n \in \mathbb{N}$.
   b) (5 pt) Prove that $2^n > n$ for all $n \in \mathbb{N}$.
   c) (5 pt) Prove that $2^{mn} > m^n$ for all $m, n \in \mathbb{N}$.
   d) (5 pt) Let X and Y be nonempty finite sets with $|X| = n$ and $|Y| = m$. How many relations are there from X to Y that are *not* functions?

3. We declare that two groups G and H are equivalent ($G \sim H$) if $G \cong H$ (G is isomorphic to H).
   a) (5 pt) Show that $\sim$ is an equivalence relation on the class of all groups.
   b) (5 pt) Show that if $G \sim H$ then G is abelian if and only if H is abelian.
   c) (5 pt) Show that if $G \sim H$ then $|G| = |H|$.
   d) (5 pt) Is the converse to part c) also true? Prove or give a counterexample.

4. (5 pt) Let $p \in \mathbb{N}$ be a prime and suppose that $|G| = p$. Show that $G \cong \mathbb{Z}_p$.

5. Let X be a set and let τ_1 and τ_2 be two topologies on X. We define $\tau_1 \bigcup \tau_2$ to be $\{A | A \in \tau_1 \text{ or } A \in \tau_2\}$ and $\tau_1 \bigcap \tau_2$ to be $\{A | A \in \tau_1 \text{ and } A \in \tau_2\}$.
   a) (5 pt) Does $\tau_1 \bigcap \tau_2$ form a topology on X? Prove or give a counterexample.
   b) (5 pt) Does $\tau_1 \bigcup \tau_2$ form a topology on X? Prove or give a counterexample.