MATH 420-620
FALL 2012
HOMEWORK 13

Due Friday November 30, 2012.

1. Let R be a commutative ring with identity and let $I \subseteq R$ be an ideal.
   a) (5 pt) Show that I is a maximal ideal if and only if R/I is a field.
   b) (5 pt) Show that I is a prime ideal if and only if R/I is an integral domain.
   c) (5 pt) Show that I is a radical ideal if and only if R/I is a reduced ring.

2. Let R be an integral domain. We say that the nonunit $\pi \in R$ is irreducible if $\pi = ab$ implies that either a or b is a unit. We say that the nonunit $0 \neq p \in R$ is a (nonzero) prime element if p divides ab implies that p divides a or p divides b.
   a) (5 pt) Show that $p \in R$ is prime if and only if (p) is a prime ideal.
   b) (5 pt) Show that any nonzero prime element is irreducible.
   c) (5 pt) Give an example of an integral domain, R, and an element $\pi \in R$ that is irreducible, but not prime.
   d) (5 pt) Show that if a can be factored into a product of primes, then this factorization is unique (up to ordering and units).

3. Let R be a PID (principal ideal domain).
   a) (5 pt) Show that every nonzero prime ideal is maximal.
   b) (5 pt) Show that R satisfies the ascending chain condition on principal ideals; that is, if you have the chain of principal ideals

$$(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots$$

   then there is an n such that $(a_{n+k}) = (a_n)$ for all $k \geq 0$.
   c) (5 pt) Show that every nonzero nonunit element of R can be factored uniquely (up to ordering and units) into prime elements.