MATH 420-620
FALL 2012
HOMEWORK 14

Due Friday December 7, 2012.

1. Let $d \neq 1$ be a square-free integer and consider the set of elements $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} | a, b \in \mathbb{Q}\}$.

   a) (5 pt) Show that $\mathbb{Q}[\sqrt{d}]$ is a field.
   b) (5 pt) Show that $\mathbb{Q}[\sqrt{d}] = \mathbb{Z}[\sqrt{d}]_S$, where S is the multiplicative set of nonzero integers.

2. (5 pt) Consider the function $\phi : \mathbb{Q}[\sqrt{d}] \longrightarrow \mathbb{Q}[\sqrt{d}]$ given by $\phi(a+b\sqrt{d}) = a - b\sqrt{d}$. Show that ϕ is an automorphism of fields.

3. Let $R := \mathbb{Z}[\sqrt{d}] = \{m+n\sqrt{d} | m, n \in \mathbb{Z}\}$ and consider the function $N : \mathbb{Z}[\sqrt{d}] \longrightarrow \mathbb{Z}$ given by $N(m + n\sqrt{d}) = m^2 - dn^2$. Show that the function N enjoys the following properties.

   a) (5 pt) $N(\alpha\beta) = N(\alpha)N(\beta)$, for all $\alpha, \beta \in R$.
   b) (5 pt) $N(\alpha) = 0$ if and only if $\alpha = 0$.
   c) (5 pt) $N(\alpha) = \pm 1$ if and only if $\alpha \in U(R)$.

4. (5 pt) Show that the Gaussian integers $\mathbb{Z}[i]$ is Euclidean. What can you say about factorization in $\mathbb{Z}[i]$?

5. Consider the integral domain $\mathbb{Z}[\sqrt{-5}]$, and $\mathbb{Z}[\sqrt{-14}]$.

   a) (5 pt) Show that the elements $2, 3, 1 + \sqrt{-5}$, and $1 - \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$
   b) (5 pt) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.
   c) (5 pt) Find an element with two irreducible factorizations of different lengths in $\mathbb{Z}[\sqrt{-14}]$.