MATH 420-620
FALL 2012
HOMEWORK 4

Due Monday September 24, 2012.

1. Let G be a group and $x, y \in G$. We say that the *commutator* of x and y (sometimes denoted $[x, y]$) is $[x, y] = x^{-1}y^{-1}xy$. Globally, we define the *commutator subgroup* of G to be

$$G' = \langle [x, y] | x, y \in G \rangle.$$

a) (5 pt) Show that the elements $x, y \in G$ commute if and only if $[x, y] = 1$.
b) (5 pt) Show that G' is a normal subgroup of G.
c) (5 pt) Show that if N is a normal subroup of G, then G/N is abelian if and only if N contains G'.

2. (5 pt) Let G be a group and $\phi : G \longrightarrow H$ be a homomorphism. Show that $\mathrm{Z}(G)$ and $\ker(\phi)$ are normal subgroups of G.

3. (5 pt) Let G be a group and H a subgroup of G such that $[G : H] = 2$. Show that $H \trianglelefteq G$. Is the same true if $[G : H] = 3$?

4. (5 pt) Let G be a group with center $\mathrm{Z}(G)$. Show that G is abelian if and only if $G/\mathrm{Z}(G)$ is cyclic.