MATH 420-620
FALL 2012
HOMEWORK 8

Due Wednesday October 24, 2012.

1. (5 pt) Let G be a finite $p-$group of order p^n. Show that for all $0 \leq k \leq n$, there is a subgroup of G of order p^k and each subgroup of order p^k is normal in a subgroup of order p^{k+1} ($k \leq n-1$).

2. (5 pt) Let p and q be distinct primes with $p < q$ and $q \not\equiv 1 \text{mod}(p)$. Show that if $|G| = pq$ then $G \cong \mathbb{Z}/pq\mathbb{Z}$.

3. (5 pt) Let G be a group. We say that G is *simple* if G contains no normal subgroups except for G itself and the identity. Show that there is no simple group of order 80.

4. Suppose that G is a group of order 72; the goal of this problem is to show that G cannot be simple.

a) (5 pt) Show that G has either 1 or 4 Sylow 3-subgroups. Conclude that is G is simple, then G must have 4 Sylow 3-subgroups.

b) (5 pt) Show that if G has 4 Sylow 3-subgroups, then the conjugation action of G on the set of Sylow 3-subgroups induces a homomorphism from $G \longrightarrow \mathrm{S}_4$.

c) (5 pt) Conclude that G must have a nontrivial normal subgroup (hint: the kernal of a homomorphism is always normal).