MATH 421-621
FALL 2013
HOMEWORK 2

Due Wednesday, January 23, 2013.

1. (5 pt) We say that the $R-$module M is *simple* if its only submodules are M and 0. Show that if M is simple and $\phi : M \longrightarrow N$ is an $R-$module homomorphism, then ϕ is either one-to-one or the 0 map.

2. Let R be a commutative ring with identity.
   a) (5 pt) Show that $\mathrm{Hom}_R(M, N)$ is an $R-$module.
   b) (5 pt) Show that if $N = M$ then $\mathrm{Hom}_R(M, N)$ also has a ring structure (is this ring necessarily commutative?).

3. (5 pt) Show that there is an $R-$module isomorphism

$$\mathrm{Hom}_R(R, M) \cong M.$$

4. Let $n > 0$ be a natural number. Compute the following.
   a) (5 pt) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.
   b) (5 pt) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$.

5. We say that the $R-$module M is *cyclic* if there is an $m \in M$ such that $M = Rm$.
   a) (5 pt) Show that if M is cyclic and $\phi : M \longrightarrow N$ is an $R-$module homomorphism, then ϕ is completely determined by $\phi(m)$.
   b) (5 pt) Show that the homomorphic image of a cyclic module is cyclic.
   c) (5 pt) Is it true that a submodule of a cyclic module is necessarily cyclic? Prove or give a counterexample.