MATH 421-621
SPRING 2013
HOMEWORK 9

Due Wednesday April 17, 2013.

1. (5 pt) Let $\mathbb{F} \subset \mathbb{K}$ be fields such that $[\mathbb{K} : \mathbb{F}]$ is prime. Show that if ξ is any element of $\mathbb{K}$ that is not in $\mathbb{F}$, then $\mathbb{K} = \mathbb{F}(\xi)$.

2. We say that the extension of fields $\mathbb{F} \subset \mathbb{K}$ is quadratic if $[\mathbb{K} : \mathbb{F}] = 2$.

a) (5 pt) Let $a, b, c \in \mathbb{F}$ with $a \neq 0$. Show that if $ax^2 + bx + c$ is irreducible, then it has a solution in some quadratic extension in $\mathbb{F}$.

b) (5 pt) Derive the solution(s) of the equation $ax^2 + bx + c = 0$. You will need to make an assumtion, what is it?

c) (5 pt) Give an example of a field with infinitely many distinct quadratic extensions.

d) (5 pt) Give an example of a field with only one quadratic extension and an example of a field with no quadratic extensions.

e) (5 pt) Give an example of a field $\mathbb{F}$ and an irreducible (over $\mathbb{F}[x]$) quadratic polynomial $ax^2 + bx + c$ that has only one root in $\mathbb{K} := \mathbb{F}[x]/(ax^2 + bx + c)$.

3. Let F_1 and F_2 be fields both contained in a larger field L and both containing K. We define the *composite* of the fields F_1 and F_2 (F_1F_2) to be the smallest subfield of L containing both F_1 and F_2.

a) (5 pt) Show that $[F_1F_2 : K] \leq [F_1 : K][F_2 : K]$.

b) (5 pt) Show that if $\gcd([F_1 : K], [F_2 : K]) = 1$ then $[F_1F_2 : K] = [F_1 : K][F_2 : K]$.