MATH 720
FALL 2003
HOMEWORK 1

Due Wednesday September 3, 2003.

1. (5 pt) Let $\mathbb{Z}$ denote the integers and let $m, n \in \mathbb{Z}$ both be nonzero. A *greatest common divisor* for m and n is an integer d such that d divides both m and n and is "greatest" in the sense that if d' divides both m and n, then d' divides d. Show that m and n have a greatest common divisor and that this greatest common divisor is a linear combination of m and n (that is, if d=gcd(m, n) then there exist $a, b \in \mathbb{Z}$ such that $d = am + bn$).

2. Let G and H be groups, $f : G \longrightarrow H$ be a homomorphism, and $x, y \in G$. We will also use the notation $|x|$ to denote the order of the element $x \in G$.

a) (5 pt) Show that $|x| = |x^{-1}| = |y^{-1}xy|$.
b) (5 pt) Show that if $|x|$ is finite then $|f(x)|$ is finite and $|f(x)|$ divides $|x|$.
c) (5 pt) Show that $|xy| = |yx|$.
d) (5 pt) Show that if $xy = yx$ then $|xy| \leq \mathrm{lcm}(|x|, |y|)$.
e) (5 pt) Does part d) hold in general? Prove that it does or give a counterexample.

3. (5 pt) Suppose that G is a *finite* group generated by two elements of order 2. Show that G is necessarily D_n for some $n \geq 2$.

4. (5 pt) Let $x \in S_n$, $n \geq 2$. Show that $|x| \leq e^{\frac{n}{e}}$. (It should be noted that this is a very naive upper bound. Hint: calculus, LaGrange Multipliers and problem 1d) might be helpful.)

5. (5 pt) Let $m \geq 2$. Show that $D_m = S_n$ if and only if $n = m = 3$. (Hint: problem number 4 might be useful.)