MATH 720
FALL 2003
HOMEWORK 2

Due Monday September 15, 2003.

1. Let H and K be subgroups of G.
   a) (5 pt) Show that HK is a subgroup of G if and only if $HK = KH$.
   b) (5 pt) Show that if either H or K is normal in G, then HK is a subgroup of G.
   c) (5 pt) Is the converse to part b) true? Prove or give a counterexample.

2. (5 pt) Let S be any subset (nonempty if you like) of a group G. Show that the set S has precisely k distinct conjugates ($\{x^{-1}Sx | x \in G\}$) in G if and only if $[G : N_G(S)] = k$. What happens in the case that S consists of a single element s?

3. (5 pt) Show that a group is abelian if and only if the map $\phi : A \longrightarrow A$ given by $\phi(a) = a^{-1}$ is an automorphism of groups.

4. Let G be a group. For any $x \in G$ we define $\phi_x : G \longrightarrow G$ by $\phi_x(y) = x^{-1}yx$. In class we showed that ϕ_x is an automorphism.
   a) (5 pt) Show that $\mathrm{Aut}(G)$ is a group.
   b) (5 pt) Show that $\mathrm{Inn}(G) = \{\phi_x | x \in G\}$ is a subgroup of $\mathrm{Aut}(G)$.
   c) (5 pt) Show that $\mathrm{Inn}(G)$ may be a proper subgroup of $\mathrm{Aut}(G)$.

5. A subgroup H of a group G is said to be *characteristic* if $\sigma(H) \subseteq H$ for all $\sigma \in \mathrm{Aut}(G)$.
   a) (5 pt) Show that any characteristic subgroup of G is normal in G.
   b) (5 pt) Show that G', the commutator subgroup of G, is characteristic (and hence normal from part a)) in G.
   c) (5 pt) Show that if $\psi : G \longrightarrow A$ is a homomorphism of groups with A abelian, then $\ker(\psi)$ contains the commutator subgroup G'.

6. (5 pt) Show that if G is a finite group and N is a normal subgroup of G such that $|N| = n$ is relatively prime to $[G : N]$ then N is the unique subgroup of G or order n.