MATH 720
FALL 2010
HOMEWORK 2

Due Friday September 17, 2010.

1. (5 pt) Let H and K be subgroups of G. Show that $HK = KH$ if and only if HK is a subgroup of G.

2. (5 pt) Let G be a finite group and H a normal subgroup of G such that $|H| = k$ is relatively prime to $[G : H]$. Show that H is the only subgroup of order k.

3. (5 pt) Show that if G is finite and H and K are subgroups such that $\gcd([G : H], [G : K]) = 1$ then $G = HK$.

4. A subgroup $H \leq G$ is said to be *characteristic* if $\sigma(H) \subseteq H$ for all $\sigma \in \mathrm{Aut}(G)$.
   a) (5 pt) Show that any characteristic subgroup of G is normal in G.
   b) (5 pt) Show that G', the commutator subgroup, is characteristic in G (and hence normal).
   c) (5 pt) Show that if $\phi : G \longrightarrow A$ is a homomorphism of groups, with A abelian, the $\ker(\phi)$ contains the commutator subgroup G'.

5. (5 pt) Let G be a finite group of order n and let p be the smallest prime dividing n. Show that if there is a subgroup $H \leq G$ such that $[G : H] = p$, then H is normal in G.