MATH 720
FALL 2003
HOMEWORK 5

Due Wednesday, October 29, 2003.

1. (5 pt) Let p and q be distinct positive primes. Show that any group of order pq or p^2q is a semidirect product of its Sylow subgroups (this problem is true in much more generality).

2. (5 pt) Give an example of a group which cannot be decomposed into a semidirect product of two of its proper subgroups.

The remainder of this assignment will be devoted to showing that the only non-abelian simple group of order less than or equal to 100 is A_5.

3. Suppose that $|G| = 60$ and that G is simple.
   a) (5 pt) Find the possibilities for the number of Sylow 2-subgroups (n) of G and show that we only need concern ourselves with $n = 5$ or $n = 15$.
   b) (5 pt) Show that if $n = 5$ then G is isomorphic to a subgroup of S_5 and conclude that $G \cong A_5$.
   c) (5 pt) Show that if $n = 15$ then there are two Sylow 2-subgroups (say P and Q) that must intersect in a subgroup of G of order 2.
   d) (5 pt) If $H = N_G(P \bigcap Q)$ show that $4 \mid |H|$ and conclude that $|H| \geq 12$. (Hint: any group of order 4 is abelian so $P \subseteq N_G(P \bigcap Q)$.)
   e) (5 pt) Show that since the index of H in G less than or equal to 5, $G \cong A_5$.

4. Assume that $|G| \leq 100$ and that G is simple and nonabelian.
   a) (5 pt) Use the results from this (and previous assignments) to make a list of the possible orders of G.
   b) (5 pt) Eliminate all possibilities except for $|G| = 60$ or 90 (most of these should be almost immediate).

5. Now assume that $|G| = 90$ and is simple.
   a) (5 pt) Show that G must have 6 Sylow 5-subgroups.
   b) (5 pt) Show that G is necessarily isomorphic to a subgroup of A_6. (Hint: G can be considered a simple subgroup of S_6; consider $G \bigcap A_6$).
   c) (5 pt) Derive a contradiction by showing that A_6 has no subgroup of order 90. (Hint: if A_6 has a subgroup of order 90, look at the orbit of this group under conjugation action...what is the order of its normalizer?)

6. (5 pt) Conclude that if $|G| \leq 100$ and G is nonabelian and simple, then $G \cong A_5$.