MATH 720
FALL 2003
HOMEWORK 6

Due Monday, November 10, 2003.

1. Let R be a ring. We say that R is of *characteristic* n if there is a positive integer n such that $na = 0$ for all $a \in R$ (if no such positive integer exists, we say that R is of characteristic 0).

a) (5 pt) Show that if R has an identity, then R is of characteristic $n > 0$ if and only if $n(1_R) = 0$ and this n is the minimal positive integer with this property.

b) (5 pt) Show that if R has no nontrivial zero divisors then the characteristic of R is prime (or 0).

c) (5 pt) Is the converse to the above statement true? Prove or give a counterexample.

d) (5 pt) We say that R is Boolean if $r^2 = r$ for all $r \in R$. Show that any Boolean ring is commutative.

e) (5 pt) Must a Boolean ring have an identity? Prove or give a counterexample.

2. (5 pt) Let R be a finite commutative ring with no nontrivial zero divisors. Show that R is a field (be careful, I am not assuming that R has an identity at the outset).

3. (5 pt) Let R be a ring. An element $a \in R$ is said to be nilpotent if there is a positive integer n such that $a^n = 0$. If R is commutative, show that the set of nilpotent elements forms an ideal of R. Give an example to show that this is not necessarily true in the noncommutative case.

4. (5 pt) (The Freshman's Dream) Let R be a commutative ring with identity and let the characteristic of R be $p > 0$ (with p prime). Show that if $a, b \in R$ then

$$(a+b)^{p^n} = a^{p^n} + b^{p^n}.$$

Use this to show that the map $\phi_n : R \longrightarrow R$ given by $\phi_n(x) = x^{p^n}$ is a ring homomorphism.

5. Let R be a (nonzero) ring with the property that for all nonzero $a \in R$, there is a unique $b \in R$ such that $aba = a$. Prove the following.

a) (5 pt) R has no nontrivial zero-divisors.

b) (5 pt) If $aba = a$ then $bab = b$.

c) (5 pt) R has an identity.

d) (5 pt) R is a division ring.