MATH 720
FALL 2003
HOMEWORK 7

Due Monday, November 17, 2003.

1. Let R be commutative with identity and $I \subseteq R$ an ideal. We define the *radical* of I to be

$$\mathrm{rad}(I) = \sqrt{I} = \{x \in R | x^n \in I \text{ for some } n\}.$$

Additionally, we say that I is radical if $I = \sqrt{I}$.

a) (5 pt) Show that $\sqrt{I}$ is an ideal of R containing I.
b) (5 pt) Show that an arbitrary intersection of prime ideals is radical.

2. Let R be commutative with identity. We say that R is zero-dimensional if every prime ideal of R is maximal.

a) (5 pt) Show that any zero-dimensional domain is a field.
b) (5 pt) Show that if R is finite, then R is zero-dimensional.
c) (5 pt) Give an example of a zero-dimensional ring that is neither finite nor a field.

3. (5 pt) Let D be a division ring. Show that in $\mathrm{M}_n(D)$ (n by n matricies over D), (0) is a maximal ideal.

4. An element $e \in R$ is said to be *idempotent* if $e^2 = e$. Suppose that e is an idempotent element that is contained in $Z(R)$.

a) (5 pt) Show that $1 - e$ is a central idempotent (if there is an identity in R).
b) (5 pt) Show that eR and $(1-e)R$ are ideals of R such that $R \cong eR \times (1-e)R$.
c) (5 pt) If R has a central idempotent, does it follow that R has an identity?

5. (5 pt) Find all prime and maximal ideals in the ring $\mathbb{Z}_k$, $k \geq 0$. Find all idempotents in the ring $\mathbb{Z}_{pq}$ where p and q are nonzero prime integers.

6. (5 pt) Consider the ring $\prod_{i=1}^{\infty} \mathbb{Z}_2$. Show that the ideals $I_n = \{\alpha = (a_1, a_2, \cdots) \in \prod_{i=1}^{\infty} \mathbb{Z}_2 | a_n = 0\}$ are all maximal ideals. Are these the only maximal ideals?

7. (5 pt) Show that $\mathbb{Z}_k$ is a principal ideal ring.