MATH 720
FALL 2010
HOMEWORK 7

Due Friday, November 19, 2010.

1. (5 pt) Show that any finite commutative ring with no nonzero zero divisors is a field.

2. We say that the commutative ring with identity, R, is zero-dimensional if every prime ideal is maximal.

a) (5 pt) Show that any zero-dimensional domain is a field.
b) (5 pt) Show that if R is finite, then R is zero dimensional.
c) (5 pt) Give an example of an infinite zero-dimensional ring that is not a field.

3. An element $e \in R$ is said to be a idempotent if $e^2 = e$ and e is said to be a central idempotent if e is idempotent and $e \in Z(R)$.

a) (5 pt) Show that if R has an identity and e is a central idempotent, then $1 - e$ is also a central idempotent.
b) (5 pt) If R has an indentity, show that eR and $(1 - e)R$ are ideals of R such that $R \cong eR \times (1 - e)R$.
c) (5 pt) If R has a central idempotent, does it follow that R has an identity?

4. Let R be a principal ideal ring (PIR) with identity (you may assume that such a ring is commutative with identity).

a) (5 pt) Show that if R is an integral domain then either R is a field or R is one-dimensional (meaning that every nonzero prime ideal is maximal).
b) (5 pt) Show that the homomorphic image of a PIR is a PIR.
c) (5 pt) Show that if $\mathfrak{P}_1 \subsetneq \mathfrak{P}_2$ are prime ideals of R, then $\mathfrak{P}_2$ is maximal (i.e. R is no more than 1-dimensional).
d) (5 pt) Is it true that any subring of a PIR is a PIR?