MATH 720
FALL 2003
HOMEWORK 8

Due Friday, December 12, 2003.

1. (5 pt) Let d be a square free integer. We define a quadratic ring of integers to be $R := \mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\}$ where

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \text{mod}(4) \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \text{mod}(4). \end{cases}$$

Show that R is a commutative ring with identity.

2. Let R be a quadratic ring of integers. We define the norm map $N : R \longrightarrow \mathbb{Z}$ by $N(a + b\omega) = (a + b\omega)(a + b\overline{\omega})$ where

$$\overline{\omega} = \begin{cases} -\sqrt{d} & \text{if } d \equiv 2, 3 \text{mod}(4) \\ \frac{1-\sqrt{d}}{2} & \text{if } d \equiv 1 \text{mod}(4). \end{cases}$$

Prove the following properties of the norm.

a) (5 pt) $N(xy) = N(x)N(y)$ for all $x, y \in R$.
b) (5 pt) $N(x) = 0$ if and only if $x = 0$.
c) (5 pt) $x \in U(R)$ if and only if $N(x) = \pm 1$.
d) (5 pt) Use the norm map to show that $\mathbb{Z}[\sqrt{-14}]$ is not an HFD.
e) (5 pt) Use the norm map to show that the ring $\mathbb{Z}[\sqrt{10}]$ is not a UFD.
f) (5 pt) Use the norm map to show that the ring $\mathbb{Z}[i]$ is Euclidean (extra credit...can you show that the ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a PID, but not Euclidean?).

3. Let R be a quadratic ring of integers.

a) (5 pt) If $d < 0$ find all of the units of R.
b) (5 pt) If $d = 2$ show that there are infinitely many units in R (actually, for all $d > 0$ there are infinitely many units).

4. (5 pt) Let R be commutative with identity. We say that R is Von Neumann regular if for all $a \in R$ there is an $x \in R$ such that $a^2x = a$. Let R be Von Neumann regular and let $\mathfrak{P}$ be a prime ideal of R. Show that $R/\mathfrak{P} \cong R_{\mathfrak{P}}$.

5. (5 pt) Show that any overring of the integers $\mathbb{Z}$ is of the form $\mathbb{Z}_S$ for some multiplicatively closed subset of $\mathbb{Z}$. (Note: an overring of a domain D, with quotient field K, is a ring R such that $D \subseteq R \subseteq K$.)

6. (5 pt) Let R be commutative with identity. Show that the following conditions are equivalent.

a) R has a unique prime ideal.
b) Every nonunit of R is nilpotent.
c) R has a minimal prime ideal which contains all zero divisors and all nonunits of R are zero-divisors.