1. (5 pt) Let $d$ be a square free integer. We define a quadratic ring of integers to be $R := \mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\}$ where

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2, 3 \mod(4) \\ \frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \mod(4). \end{cases}$$

Show that $R$ is a commutative ring with identity.

2. Let $R$ be a quadratic ring of integers. We define the norm map $N : R \rightarrow \mathbb{Z}$ by $N(a + b\omega) = (a + b\omega)(a + b\overline{\omega})$ where

$$\overline{\omega} = \begin{cases} -\sqrt{d} & \text{if } d \equiv 2, 3 \mod(4) \\ \frac{1 - \sqrt{d}}{2} & \text{if } d \equiv 1 \mod(4). \end{cases}$$

Prove the following properties of the norm.

a) (5 pt) $N(xy) = N(x)N(y)$ for all $x, y \in R$.

b) (5 pt) $N(x) = 0$ if and only if $x = 0$.

c) (5 pt) $x \in U(R)$ if and only if $N(x) = \pm 1$.

d) (5 pt) Use the norm map to show that $\mathbb{Z}[\sqrt{-14}]$ is not an HFD.

e) (5 pt) Use the norm map to show that the ring $\mathbb{Z}[\sqrt{10}]$ is not a UFD.

f) (5 pt) Use the norm map to show that the ring $\mathbb{Z}[\frac{1 + \sqrt{-19}}{2}]$ is a PID, but not Euclidean? (extra credit...can you show that the ring $\mathbb{Z}[\sqrt{14}]$ is not a PID?)

3. Let $R$ be a quadratic ring of integers.

a) (5 pt) If $d < 0$ find all of the units of $R$.

b) (5 pt) If $d = 2$ show that there are infinitely many units in $R$ (actually, for all $d > 0$ there are infinitely many units).

4. (5 pt) Let $R$ be commutative with identity. We say that $R$ is Von Neumann regular if for all $a \in R$ there is an $x \in R$ such that $a^2x = a$. Let $R$ be Von Neumann regular and let $\mathfrak{P}$ be a prime ideal of $R$. Show that $R/\mathfrak{P} \cong R_{\mathfrak{P}}$.

5. (5 pt) Show that any overring of the integers $\mathbb{Z}$ is of the form $\mathbb{Z}[S]$ for some multiplicatively closed subset of $\mathbb{Z}$. (Note: an overring of a domain $D$, with quotient field $K$, is a ring $R$ such that $D \subseteq R \subseteq K$.)

6. (5 pt) Let $R$ be commutative with identity. Show that the following conditions are equivalent.

a) $R$ has a unique prime ideal.

b) Every nonunit of $R$ is nilpotent.

c) $R$ has a minimal prime ideal which contains all zero divisors and all nonunits of $R$ are zero-divisors.