MATH 720
FALL 2010
EXAM 1

Due Monday October 11, 2010.

1. (5 pt) Show that every finitely-generated subgroup of $\mathbb{Q}$ is cyclic.

2. (5 pt) Let F be free on the set X and $n \in \mathbb{N}$. Show that the subgroup of F generated by the set $\{g^n | g \in F\}$ is normal in F.

3. (5 pt) Let G be a group. Show that $\mathrm{Inn}(G) \cong G/Z(G)$.

4. Let G be a group with center $Z(G)$, and p a positive prime integer.
   a) (5 pt) Show that if $G/Z(G)$ is cyclic, then G is abelian.
   b) (5 pt) Use this to show that if $|G| = p^2$, then G is abelian.
   c) (5 pt) Show that if $|G| = p^3$ then

$$Z(G) \cong \begin{cases} G & \text{if } G \text{ is abelian} \\ \mathbb{Z}_p & \text{if } G \text{ is not abelian} \end{cases}$$

   d) (5 pt) Show that if $|G| = p^3$ and G is not abelian, then $G/Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.