MATH 720
FALL 2010
EXAM 2

Due Monday November 29, 2010.

1. Let S be a subset of a commutative ring with identity, R. We say that S is multiplicatively closed if $s, t \in S$ implies that $st \in S$.

a) (5 pt) Let S be a multiplicatively closed subset of R and I an ideal of R such that $I \bigcap S = \emptyset$. Show that there is an ideal $J \supseteq I$ that is maximal with respect to the property that $J \bigcap S = \emptyset$ (that is, any ideal containing J must have nonempty intersection with S). We say that the ideal J is maximal with respect to the exclusion of S.

b) (5 pt) Show if J is maximal with respect to the exclusion of S, then J is prime.

c) (5 pt) Take the specific case of S being the units of R and I any proper ideal of R. Use the above results to conclude that I is contained in a maximal ideal of R.

2. In this problem we will characterize $\mathrm{rad}(I)$. For this problem R is a commutative ring with identity and $I \subsetneq R$ is a proper ideal. Additionally we define $\mathrm{N}(R)$ to be the ideal consisting of all nilpotent elements of R.

a) (5 pt) Show that $\mathrm{N}(R) \subseteq \bigcap_{\mathfrak{P}\text{: prime}} \mathfrak{P}$.

b) (5 pt) Show that $\bigcap_{\mathfrak{P}\text{: prime}} \mathfrak{P} \subseteq \mathrm{N}(R)$. (Hint: for this part, assume that there is an element x in the intersection of all primes that is not nilpotent. Now consider the multiplicatively closed set $\{x^n | n \geq 0\}$. By the above, you should be able to expand (0) to a prime ideal that is maximal with respect to the exclusion of this set. Derive a contradiction.)

c) (5 pt) Now let I be an arbitrary ideal of a commutative ring with identity, R. Show that

$$\mathrm{rad}(I) = \bigcap_{I \subseteq \mathfrak{P}\text{: prime}} \mathfrak{P}.$$

3. (5 pt) Let R be a commutative ring with identity. Show that the set of all zero divisors of R must contain at least one prime ideal of R.

4. (5 pt) An integral domain is called one-dimensional if (0) is not a maximal ideal and every nonzero prime ideal is maximal. Show that any PID that is not a field is one-dimensional.

5. Let R be a ring.

a) (5 pt) If $a \in R$, show that $\{r \in R | ra = 0\}$ is a left ideal of R (called the left annihilator of a).

b) (5 pt) If I is a left ideal of R then show that the set $\{r \in R | rx = 0, \ \forall x \in I\}$ is an ideal of R.

c) (5 pt) If I is an ideal of R, show that $[I : R] = \{r \in R | xr \in I, \ \forall x \in R\}$ is an ideal of R.