MATH 720
FALL 2003
FINAL EXAM

Due Friday, December 19, 2003. The usual rules apply...I am the only biological resource that you should use for help.

1. (5 pt) Let R be a PID. Show that $R[[x]]$ is a UFD. (Hint: you can eliminate the case where R is a field (why?). Then show that if $\mathfrak{P}$ is a nonzero prime ideal of $R[[x]]$ then either $x \in \mathfrak{P}$ or if not then there is a suitable nonzero prime element contained in $\mathfrak{P}$).

2. Let R be an integral domain. We define a function $d : R[x] \times R[x] \longrightarrow \mathbb{Q}$ by $d(f(x), g(x)) = 2^{-n}$ where $f(x) - g(x) = x^n h(x),\ \ h(0) \neq 0$ (and we declare that $d(f(x), f(x)) = 0$).

a) (5 pt) Show that d is a metric on $R[x]$.

b) (5 pt) Describe a Cauchy sequence with respect to this metric on $R[x]$ and find the completion of $R[x]$ with respect to this metric.

3. Let R be commutative with identity.

a) (5 pt) Show that if there is an ideal $I \subseteq R$ that is not finitely generated, then there is an ideal $J \subseteq R$ that is maximal with respect to the property of not being finitely generated.

b) (5 pt) Show that any ideal that is maximal with respect to the property of not being finitely generated is prime.

c) (5 pt) Use this to show that a ring is Noetherian if and only if every *prime* ideal is finitely generated.

4. (5 pt) Use the result of the previous problem to show that if R is Noetherian, then so is $R[[x]]$. (Hint: if Γ is a prime of $R[[x]]$, consider the ideal $\{f(0) | f \in \Gamma\} \subseteq R$; this must be finitely generated as R is Noetherian.)