MATH 720
FALL 2010
FINAL EXAM

Due Wednesday, December 15, 2010.

1. For this problem, R will be a UFD with quotient field K.
   a) (5 pt) Show that $K[x]$ is a PID.
   b) (5 pt) Show that if π is a prime element of R, then π is also a prime element of $R[x]$.
   c) (5 pt) Show that if $\mathfrak{P} \subseteq R[x]$ is a prime ideal then $\mathfrak{P} \bigcap R$ is a prime ideal of R.
   d) (5 pt) Show that $R[x]$ is a UFD (hint: show every nonzero prime ideal of $R[x]$ contains a nonzero prime element by considering the cases when $\mathfrak{P} \bigcap R$ is zero and nonzero).

3. Let R be commutative with identity.
   a) (5 pt) Show that if there is an ideal $I \subseteq R$ that is not finitely generated, then there is an ideal $J \subseteq R$ that is maximal with respect to the property of not being finitely generated.
   b) (5 pt) Show that any ideal that is maximal with respect to the property of not being finitely generated is prime.
   c) (5 pt) Use this to show that a ring is Noetherian if and only if every *prime* ideal is finitely generated.

4. (5 pt) Suppose that R is commutative with identity. Show that if R is Noetherian, then $R[x]$ is Noetherian.