MATH 721
SPRING 2004
HOMEWORK 1

Due Friday January 23, 2004.

1. (5 pt) Let R be a commutative ring with identity and M some $R-$module. Show that (as $R-$modules) there is an isomorphism

$$\mathrm{Hom}_R(R, M) \cong M.$$

2. Let R be a commutative ring with 1, M an $R-$module and $m \in M$.
   a) (5 pt) Show that $I_m = \{r \in R | rm = 0\}$ is an ideal of R. (If $I_m \neq 0$ then we say that m is a torsion element of M.)
   b) (5 pt) If R is a domain, then show that the set of torsion elements of M forms a submodule of M (and show that this may not be the case if R is not a domain).

3. (5 pt) *Is it only size that matters?* Consider the $\mathbb{Z}-$modules, $\mathbb{Q}$ and $\oplus_{i=1}^{\infty}\mathbb{Z}$. Both of these modules are torsion-free and of the same cardinality. Are they isomorphic?

4. (5 pt) Let $f : A \longrightarrow B$ be an $R-$module homomorphism. Show f is an isomorphism if and only if there is an $R-$module homomorphism $g : B \longrightarrow A$ such that $fg = 1_B$ and $gf = 1_A$. Are both these conditions necessary?

5. (5 pt) Let $f : A \longrightarrow A$ be an $R-$module homomorphism such that $f(f(x)) = f(x)$ for all $x \in A$. Show that $A \cong \ker(f) \oplus \mathrm{Im}(f)$. Give a nontrivial (i.e., f not the identity or the 0 map) example of such an f and show how the decomposition works.