MATH 721
SPRING 2004
HOMEWORK 2

Due Monday February 2, 2004.

1. (*The Five Lemma.*) Consider the following commutative diagram of $R-$module homomorphisms with exact rows

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_4} & & \downarrow{\scriptstyle g_5} \\ B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5 \end{array}$$

a) (5 pt) Show that if g_2 and g_4 are onto and g_5 is one to one then g_3 is onto.
b) (5 pt) Show that if g_2 and g_4 are one to one and g_1 is onto then g_3 is one to one.
c) (5 pt) Establish the short five lemma as a special case of a) and b).

2. (*The 3×3 Lemma.*) Consider the following commutative diagram of $R-$module homomorphisms

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

a) (5 pt) Show that if the columns and the bottom two rows are exact, then the top row is exact.
b) (5 pt) Show that if the columns and the top two rows are exact, then the bottom row is exact.

3. (*The Snake Lemma.*) Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & & \end{array}$$

a) (5 pt) Show that there is an exact sequence

$$\ker(g_1) \xrightarrow{\alpha_1} \ker(g_2) \xrightarrow{\alpha_2} \ker(g_3) \xrightarrow{\partial} \operatorname{coker}(g_1) \xrightarrow{\beta_1} \operatorname{coker}(g_2) \xrightarrow{\beta_2} \operatorname{coker}(g_3)$$

b) (5 pt) Show that if f_1 is one to one, then so is α_1.

c) (5 pt) Show that if h_2 is onto, then so is β_2.

4. An $R-$module S is said to be simple if the only submodules of S are itself and 0.
   a) (5 pt) Show that any simple $R-$ module is cyclic (that is, is of the form Ra for some $a \in S$).
   b) (5 pt) Characterize all $R-$ module homomorphisms $f : S \longrightarrow S$ where S is a simple $R-$module.

5. (5 pt) If $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are $R-$module homomorphisms such that $gf = 1_A$ then $B \cong A \oplus \ker(g)$.