1. Let $D$ be a divisible abelian group.
   a) (5 pt) Show that any homomorphic image of $D$ is again divisible.
   b) (5 pt) Show that if $G$ is any abelian group, then $G \cong D_1 \oplus N$ where $D_1$ is a
divisible abelian group and $N$ is an abelian group with no nontrivial divisible
subgroups.
   c) (5 pt) Show that any abelian group may be embedded in a divisible abelian
group (more generally, any unitary $R$–module $M$ may be embedded in the
injective $R$–module $\text{Hom}_\mathbb{Z}(R, D)$ where $D$ is a divisible group containing $M$).

2. (5 pt) Show that the following conditions on an $R$–module $I$ are equivalent.
   a) $I$ is injective.
   b) Every short exact sequence of the form
      $0 \rightarrow I \rightarrow B \rightarrow C \rightarrow 0$
is split exact.
   c) $I$ is a direct summand of any module of which it is a submodule.

3. (5 pt) Let $R$ be a ring. Show that as a ring $\text{Hom}_R(R, R) \cong R^{\text{op}}$ (here $R^{\text{op}}$ is
the opposite ring of $R$ where the underlying set is the same and the multiplication is
given by $r \circ s = sr$).

4. Let $A$ be an abelian group, show that we have the following isomorphisms of
abelian groups.
   a) (5 pt) $\text{Hom}_\mathbb{Z}(\mathbb{Z}_m, A) \cong A[m]$ where $A[m] = \{a \in A | ma = 0\}$.
   b) (5 pt) $\text{Hom}_\mathbb{Z}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z},$ where $d = \gcd(m, n)$.

5. (5 pt) Let $\{J_i\}_{i \in I}$ be a family of $R$–modules. Show that $\prod_{i \in I} J_i$ is injective if and
only if $J_i$ is injective for all $i \in I$. 