MATH 721
SPRING 2004
HOMEWORK 3

Due Wednesday February 11, 2004.

1. Let D be a divisible abelian group.
   a) (5 pt) Show that any homomorphic image of D is again divisible.
   b) (5 pt) Show that if G is any abelian group, then $G \cong D_1 \oplus N$ where D_1 is a divisible abelian group and N is an abelian group with no nontrivial divisible subgroups.
   c) (5 pt) Show that any abelian group may be embedded in a divisible abelian group (more generally, any unitary $R-$module M may be embedded in the injective $R-$module $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ where D is a divisible group containing M).

2. (5 pt) Show that the following conditions on an $R-$module I are equivalent.
   a) I is injective.
   b) Every short exact sequence of the form
   $$0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$$
   is split exact.
   c) I is a direct summand of any module of which it is a submodule.

3. (5 pt) Let R be a ring. Show that *as a ring* $\mathrm{Hom}_R(R, R) \cong R^{\mathrm{op}}$ (here R^{op} is the opposite ring of R where the underlying set is the same and the multiplication is given by $r \circ s = sr$).

4. Let A be an abelian group, show that we have the following isomorphisms of abelian groups.
   a) (5 pt) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m]$ where $A[m] = \{a \in A | ma = 0\}$.
   b) (5 pt) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where $d = \gcd(m, n)$.

5. (5 pt) Let $\{J_i\}_{i \in I}$ be a family of $R-$modules. Show that $\prod_{i \in I} J_i$ is injective if and only if J_i is injective for all $i \in I$.