MATH 721
SPRING 2011
HOMEWORK 3

Due Wednesday February 16, 2010.

1. (15 pt) Show that the following conditions on an $R-$module I are equivalent.
   a) I is injective.
   b) Every short exact sequence of the form
   $$0 \longrightarrow I \longrightarrow B \longrightarrow C \longrightarrow 0$$
   is split exact.
   c) I is a direct summand of any module of which it is a submodule.

2. (10 pt) Show that the sequence of $R-$module homomorphisms
$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
is exact if and only if for every $R-$module D, the sequence of $R-$module homomorphisms
$$0 \longrightarrow \mathrm{Hom}_R(C, D) \xrightarrow{\overline{g}} \mathrm{Hom}_R(B, D) \xrightarrow{\overline{f}} \mathrm{Hom}_R(A, D)$$

3. Let A be an abelian group, show that we have the following isomorphisms of abelian groups.
   a) (5 pt) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m]$ where $A[m] = \{a \in A | ma = 0\}$.
   b) (5 pt) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where $d = \gcd(m, n)$.

4. (5 pt) Let $\{J_i\}_{i \in I}$ be a family of $R-$modules. Show that $\prod_{i \in I} J_i$ is injective if and only if J_i is injective for all $i \in I$.