MATH 721
SPRING 2004
HOMEWORK 4

Due Monday March 8, 2004.

1. Find the canonical forms (the rational canonical form, primary rational canonical form and Jordan canonical form if possible) for the following matrices over $\mathbb{Q}$:

a) (5 pt) $\begin{bmatrix} -1 & 2 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

b) (5 pt) $\begin{bmatrix} 3 & 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & -2 & 4 & 2 & 0 \\ 0 & -1 & 0 & 3 & 0 \\ 1 & -1 & 0 & -1 & 3 \end{bmatrix}$

2. (5 pt) Show that an $n \times n$ matrix (A) over a field $\mathbb{F}$ is similar to a diagonal matrix if and only if there is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

3. A matrix A is said to be nilpotent if there is an $m \geq 1$ such that $A^m = 0$. Additionally, we define the trace of A ($\mathrm{tr}(A)$) to be the sum of the diagonal elements of A. For this problem, you may assume that A is an $n \times n$ matrix over a field $\mathbb{F}$.

a) (5 pt) Show that if $A^n \neq 0$ then A is not nilpotent (i.e. if A is nilpotent then a power less than or equal to n must annihilate the matrix).
b) (5 pt) Show that if P is an invertible $n \times n$ matrix then $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A)$.
c) (5 pt) Show that A is nilpotent if and only if all of its eigenvalues are 0 (you may assume that all of the eigenvalues are in the field).
d) (5 pt) Show that if A is nilpotent, then $\mathrm{tr}(A) = 0$.
e) (5 pt) Determine the status of the converse of the statement in part d).