MATH 721
SPRING 2004
HOMEWORK 5

Due Monday March 22, 2004.

1. (5 pt) Let $K \subseteq F$ be fields. We define $\mathrm{Aut}_K(F)$ to be the set of automophisms $\sigma : F \longrightarrow F$ such that $\sigma(k) = k$ for all $k \in K$. This is called the *Galois group* of F over K. Show that if $[F : K] = n$ then $|\mathrm{Aut}_K(F)|$ divides n. What happens when n is prime?

2. Compute the following Galois groups.
   a) (5 pt) $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$ (hint: first show that any element of $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$ is continuous).
   b) (5 pt) $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{d}))$ where d is a square-free integer.
   c) (5 pt) $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}))$.
   d) (5 pt) $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2}, \sqrt{3}))$.

3. (5 pt) Show that if $K \subseteq F$ and $u \in F$ is algebraic over K of odd degree, then $K(u^2) = K(u)$.

4. (5 pt) Show that if F is algebraic over K, and D is an integral domain such that $K \subseteq D \subseteq F$, then D is a field.

5. (5 pt) Let F be a finite field of characteristic p. Show that $\mathrm{Aut}_{\mathbb{Z}_p}(F)$ is cyclic.

6. (5 pt) Let $K \subseteq F$ be fields and let $\overline{K}_F = \{z \in F | z \text{ is algebraic over K}\}$. Show that $\overline{K}_F$ is a subfield of F containing K (this is called the algebraic closure of K in F).