MATH 721
SPRING 2011
EXAM 1

Due Monday February 28, 2011. As is usual on exams, I am the only biological resource that you should use.

1. (*Localizations are flat*) Let R be an integral domain, let N be an $R-$module and $S \subseteq R$ a multiplicative set.

a) (5 pt) Show that the set $S^{-1}N = \{\frac{n}{s} | n \in N, s \in S\}$ (where we declare that $\frac{n_1}{s_1} = \frac{n_2}{s_2}$ if and only if there is a $t \in S$ such that $t(s_2n_1 - s_1n_2)$=0) is an $R-$module with addition given by

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{s_2n_1 + s_1n_2}{s_1s_2}$$

and $R-$ action

$$r(\frac{n}{s}) = \frac{rn}{s}.$$

b) (5 pt) Show that every element of $R_S \otimes_R N$ can be written as a single tensor of the form $(\frac{1}{s}) \otimes n$ with $s \in S$ and $n \in N$.

c) (5 pt) Show that there is an isomorphism $R_S \otimes_R N \cong S^{-1}N$.

d) (5 pt) Show that a tensor of the form $(\frac{1}{s}) \otimes n$ is 0 if and only if there exists $t \in S$ such that $tn = 0$.

e) (5 pt) Show that R_S is a flat $R-$module.

f) (5 pt) Show that $\mathbb{Q}$ is a flat $\mathbb{Z}-$module that is not projective.

2. Let R be an integral domain with quotient field K and M an $R-$module. Let $T(M) = \{x \in M | rx = 0 \text{ for some nonzero } r \in R\}$. We have shown that $T(M)$ is a submodule of M.

a) (5 pt) Show that $M/T(M)$ is torsion free.

b) (5 pt) Show that $M \otimes_R K \cong M/T(M) \otimes_R K$.

3. (5 pt) We showed in class that tensor product and direct sum commute (that is, $M \otimes_R (\oplus_{i \in I} A_i) \cong \oplus_{i \in I}(M \otimes_R A_i)$). Does this result hold in general for direct products (i.e. is $M \otimes_R (\prod_{i \in I} A_i) \cong \prod_{i \in I}(M \otimes_R A_i)$)?

4. (5 pt) (*Adjoint Associativity*) Show that if R is commutative with identity and A, B, and C are $R-$modules, then we have an $R-$module isomorphism

$$\text{Hom}_R(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_R(B, C)).$$

5. Suppose that R is commutative with identity, $I \subseteq R$ an ideal, and J and M are $R-$modules.

a) (5 pt) Show that $R/I \otimes_R M \cong M/IM$.

b) (5 pt) Show that if $R = \mathbb{Z}$ and J is injective, then $M \otimes_R J$ is injective.