Due Monday February 28, 2011. As is usual on exams, I am the only biological resource that you should use.

1. (Localizations are flat) Let \( R \) be an integral domain, let \( N \) be an \( R \)-module and \( S \subseteq R \) a multiplicative set.
   a) (5 pt) Show that the set \( S^{-1}N = \{ \frac{n}{s} | n \in N, s \in S \} \) (where we declare that \( \frac{n}{s_1} = \frac{n}{s_2} \) if and only if there is a \( t \in S \) such that \( t(s_2n_1 - s_1n_2) = 0 \)) is an \( R \)-module with addition given by
      \[
      \frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{s_2n_1 + s_1n_2}{s_1s_2}
      \]
      and \( R \)-action
      \[
      r(\frac{n}{s}) = \frac{rn}{s}.
      \]
   b) (5 pt) Show that every element of \( R_S \otimes_R N \) can be written as a single tensor of the form \( (\frac{1}{s}) \otimes n \) with \( s \in S \) and \( n \in N \).
   c) (5 pt) Show that there is an isomorphism \( R_S \otimes_R N \cong S^{-1}N \).
   d) (5 pt) Show that a tensor of the form \( (\frac{1}{s}) \otimes n \) is 0 if and only if there exists \( t \in S \) such that \( tn = 0 \).
   e) (5 pt) Show that \( R_S \) is a flat \( R \)-module.
   f) (5 pt) Show that \( \mathbb{Q} \) is a flat \( \mathbb{Z} \)-module that is not projective.

2. Let \( R \) be an integral domain with quotient field \( K \) and \( M \) an \( R \)-module. Let \( T(M) = \{ x \in M | rx = 0 \text{ for some nonzero } r \in R \} \). We have shown that \( T(M) \) is a submodule of \( M \).
   a) (5 pt) Show that \( M/T(M) \) is torsion free.
   b) (5 pt) Show that \( M \otimes_R K \cong M/T(M) \otimes_R K \).

3. (5 pt) We showed in class that tensor product and direct sum commute (that is, \( M \otimes_R (\oplus_{i \in I} A_i) \cong \oplus_{i \in I} (M \otimes_R A_i) \)). Does this result hold in general for direct products (i.e. is \( M \otimes_R (\prod_{i \in I} A_i) \cong \prod_{i \in I} (M \otimes_R A_i) \))?

4. (5 pt) (Adjoint Associativity) Show that if \( R \) is commutative with identity and \( A, B, \) and \( C \) are \( R \)-modules, then we have an \( R \)-module isomorphism
   \[
   \text{Hom}_R(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_R(B, C)).
   \]

5. Suppose that \( R \) is commutative with identity, \( I \subseteq R \) an ideal, and \( J \) and \( M \) are \( R \)-modules.
   a) (5 pt) Show that \( R/I \otimes_R M \cong M/IM \).
   b) (5 pt) Show that if \( R = \mathbb{Z} \) and \( J \) is injective, then \( M \otimes_R J \) is injective.