MATH 721
SPRING 2004
EXAM 2

Due Monday April 5, 2004. As is usual on exams, I am the only biological resource that you should use.

1. Consider the polynomial $f(x) = x^5 - 3 \in \mathbb{Q}[x]$.

   a) (5 pt) Let F be the field obtained by adjoining one of the complex roots of the above polynomial to $\mathbb{Q}$. Find $[F : \mathbb{Q}]$ and show that this extension is not Galois.

   b) (5 pt) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible over F.

   c) (5 pt) If $\overline{F}$ is the field obtained by adjoining all of the roots of $f(x)$ to $\mathbb{Q}$, find the Galois group $\mathrm{Gal}(\overline{F}/\mathbb{Q})$. (Hint: this group must be a transitive subgroup of S_5.)

2. Consider the field, F, obtained by adjoining all roots of the polynomial $f(x) = x^6 - 4x^3 + 1$ to the rational numbers $\mathbb{Q}$.

   a) (5 pt) Show that complex conjugation is a nontrivial automorphism of F.

   b) (5 pt) Show that $[F : \mathbb{Q}] = 12$.

   c) (5 pt) If α is a real root of this polynomial, show that the map induced by $\alpha \mapsto \alpha^{-1}$ gives rise to an automorphism of $\mathbb{Q}(\alpha)$.

   d) (5 pt) Find the Galois group of F over $\mathbb{Q}$.

3. (5 pt) Consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(x)$. Show that $\mathbb{Q}(x^2)$ is a closed intermediate extension, but $\mathbb{Q}(x^3)$ is not.

4. (5 pt) Show that if K is a field such that $\mathrm{char}(K) \neq 2$ and $[F : K] = 2$ then F is Galois over K.

5. (5 pt) If E is Galois over K and F is Galois over E, is it true that F is Galois over K? Prove the statement or give a counterexample.