Due Monday April 5, 2004. As is usual on exams, I am the only biological resource that you should use.

1. Consider the polynomial $f(x) = x^5 - 3 \in \mathbb{Q}[x]$.
   a) (5 pt) Let $F$ be the field obtained by adjoining one of the complex roots of the above polynomial to $\mathbb{Q}$. Find $[F : \mathbb{Q}]$ and show that this extension is not Galois.
   b) (5 pt) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible over $F$.
   c) (5 pt) If $\overline{F}$ is the field obtained by adjoining all of the roots of $f(x)$ to $\mathbb{Q}$, find the Galois group $\text{Gal}(\overline{F}/\mathbb{Q})$. (Hint: this group must be a transitive subgroup of $S_5$.)

2. Consider the field, $F$, obtained by adjoining all roots of the polynomial $f(x) = x^6 - 4x^3 + 1$ to the rational numbers $\mathbb{Q}$.
   a) (5 pt) Show that complex conjugation is a nontrivial automorphism of $F$.
   b) (5 pt) Show that $[F : \mathbb{Q}] = 12$.
   c) (5 pt) If $\alpha$ is a real root of this polynomial, show that the map induced by $\alpha \mapsto \alpha^{-1}$ gives rise to an automorphism of $\mathbb{Q}(\alpha)$.
   d) (5 pt) Find the Galois group of $F$ over $\mathbb{Q}$.

3. (5 pt) Consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(x)$. Show that $\mathbb{Q}(x^2)$ is a closed intermediate extension, but $\mathbb{Q}(x^3)$ is not.

4. (5 pt) Show that if $K$ is a field such that $\text{char}(K) \neq 2$ and $[F : K] = 2$ then $F$ is Galois over $K$.

5. (5 pt) If $E$ is Galois over $K$ and $F$ is Galois over $E$, is it true that $F$ is Galois over $K$? Prove the statement or give a counterexample.