MATH 721
SPRING 2011
FINAL EXAM

Due before I have to chase you to Pakistan. As is usual on exams, I am the only biological resource that you should use.

1. Let R be a domain with quotient field K and $\overline{R}$ the integral closure of R. V will denote a valuation overring of R.

   a) Show that $\overline{R}$ is integrally closed.
   b) Show that $\overline{R} \subseteq \bigcap_{R \subseteq V \subseteq K} V$.
   c) Now show that $\overline{R} = \bigcap_{R \subseteq V \subseteq K} V$.

2. Let R be a domain with quotient field K. An element $x \in K$ is said to be *almost integral* if there is a nonzero $r \in R$ such that $rx^n \in R$ for all $n \in \mathbb{N}$. We say that a domain is *completely integrally closed* if it contains all of its almost integral elements.

   a) Show that if $x \in K$ is integral over R, then x is almost integral over R.
   b) Give an example of an element that is almost integral, but not integral.
   c) Show that if R is Noetherian, then any almost integral element over R is integral over R.
   d) Let V be a valuation domain that is not a field. Show that V is completely integrally closed if and only if V is one-dimensional (that is, every nonzero prime ideal is maximal).

3. We say that R is a Prüfer domain if $R_{\mathfrak{P}}$ is a valuation domain for all prime ideals $\mathfrak{P} \subseteq R$. We say that R is a Bezout domain if every finitely generated ideal is principal.

   a) Show that V is a valuation domain if and only if V is a quasilocal Bezout domain.
   b) Show that R is a Prüfer domain if and only if every finitely generated ideal of R is invertible.