MATH 724
FALL 2005
HOMEWORK 1

Due Monday September 5, 2005.

1. Let R be a commutative ring with identity and $\mathfrak{N} \subseteq R$ the set of nilpotent elements of R.

a) (5 pt) Show that $\mathfrak{N}$ is an ideal.

b) (5 pt) Show that

$$\mathfrak{N} = \bigcap_{\mathfrak{P}:\text{prime}} \mathfrak{P}.$$

c) (5 pt) Generalize part b) and show that if $I \subseteq R$ is an ideal, then

$$\sqrt{I} = \bigcap_{\mathfrak{P} \supseteq I} \mathfrak{P},$$

where the intersection is taken over all prime ideals containing I.

2. (5 pt) Let R be a commutative ring with identity and $I, J \subseteq R$ ideals. We define $(I : J) = \{r \in R | rJ \subseteq I\}$. Show that $(I : J)$ is an ideal of R.

3. Let R be a commutative ring with identity.

a) (5 pt) Show that if R contains a non-principal ideal, then R contains an ideal, I, that is maximal with respect to being non-principal.

b) (5 pt) Show that any ideal that is maximal with respect to being non-principal is prime.

c) (5 pt) Show that if R contains an ideal that is not finitely generated, then R contains an ideal, J, that is maximal with respect to being not finitely generated.

d) (5 pt) Show that any ideal that is maximal with respect to being not finitely generated is prime.

e) (5 pt) Show that R is a PID if and only if every prime ideal of R is principal.

f) (5 pt) Show that R is Noetherian (every ideal is finitely generated) if and only if every prime ideal is finitely generated.

4. (5 pt) Let R be a integral domain. Show that the following conditions are equivalent.

(1) Every $R-$module is projective.
(2) Every $R-$module is injective.
(3) Every $R-$module is free.
(4) R is a field.

Note that 1)$\Longleftrightarrow$ 2) and does not require the assumption integral domain, can you give an example of a commutative ring with 1 such that every ideal is projective (and injective) but not every module is free?