1. Let $R$ be a domain, and $a, b \in R$. We define the greatest common divisor of $a$ and $b$ to be a common divisor $d := \gcd(a, b)$ with the property that if $x$ is any other common divisor of $a$ and $b$ then $x$ divides $d$. We also define the least common multiple to be a common multiple $L := \text{lcm}(a, b)$ with the property that if $y$ is any other common multiple of $a$ and $b$ then $L$ divides $y$.

a) (5 pt) Give an example of a domain, $R$, and two elements $a, b \in R$ such that $\gcd(a, b)$ exists but $\text{lcm}(a, b)$ does not.

b) (5 pt) Show that if $\text{lcm}(a, b)$ exists, then so does $\gcd(a, b)$.

2. A GCD domain is an integral domain in which every two (nonzero) elements have a greatest common divisor. Show that in a GCD domain the following hold.

a) (5 pt) $x(\gcd(a, b)) = \gcd(xa, xb)$.

b) (5 pt) If $\gcd(a, b) = d$ then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.

c) (5 pt) If $\gcd(x, a) = 1$ and $\gcd(x, b) = 1$ then $\gcd(x, ab) = 1$.

d) (5 pt) If $\gcd(x, a) = 1$ and $x$ divides $ab$ then $x$ divides $b$.

e) (5 pt) Which of the above hold in a general integral domain?

3. A Bezout domain is a domain where every finitely generated ideal is principal.

a) (5 pt) Show $R$ is a valuation domain if and only if $R$ is a quasi-local Bezout domain.

b) (5 pt) Show that any Bezout domain is a GCD domain.

c) (5 pt) Show that any UFD is a GCD domain.

d) (5 pt) Give examples to show that the classes of Bezout domains and UFDs are distinct.