1. Let \( R \) be an integral domain. A nonzero nonunit element \( z \in R \) is said to be a universal side divisor if given any \( x \in R \) there is a \( r \in R \) such that
\[
x = rz + v
\]
where \( v \) is either 0 or a unit in \( R \). Let \( R \) be a Euclidean domain with norm function \( \phi \).
   a) (5 pt) Show that any nonunit in \( R \) of minimal norm is a universal side divisor.
   b) (5 pt) Show that \( \mathbb{Z}[\frac{1+\sqrt{-19}}{2}] \) is not Euclidean.

2. Let \( d \) be a squarefree integer. We define
\[
R = \mathbb{Z}[\omega] \quad \text{where} \quad \omega = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \mod(4); \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \mod(4). \end{cases}
\]
a) (5 pt) Show that \( R \) is integral over \( \mathbb{Z} \).
   b) (5 pt) We define a norm to be a map \( N : R \rightarrow \mathbb{N}_0 \) satisfying \( N(0) = 0 \) and \( N(ab) = N(a)N(b) \). Show that \( N : \mathbb{Z}[\omega] \rightarrow \mathbb{N}_0 \) defined by \( N(a+b\omega) = (a+b\omega)(a+b\overline{\omega}) \) is a norm.
   c) (5 pt) Use the norm to show that \( \mathbb{Z}[\omega] \) is atomic.
   d) (5 pt) Show that the ring \( \mathbb{Z}[\sqrt{-14}] \) is not a UFD.

3. Let \( R \) be a domain and \( N \) a norm on \( R \). We say that \( N \) is a Dedekind-Hasse norm if \( N \) is positive and for every nonzero \( x, y \in R \) either \( y \) is divisible by \( x \) or we can find \( a, b \in R \) such that
\[
0 < N(ax+y) < N(x).
\]
a) (5 pt) Show that \( R \) is a PID if and only if \( R \) has a Dedekind-Hasse norm.
   b) (5 pt) Show that the norm defined in problem 2 for the ring \( \mathbb{Z}[\frac{1+\sqrt{-19}}{2}] \) is a Dedekind-Hasse norm (hence \( \mathbb{Z}[\frac{1+\sqrt{-19}}{2}] \) is a PID that is not Euclidean).

4. Suppose that \( R \) is a UFD.
   a) (5 pt) Show that \( R[[x]] \) is atomic.
   b) (5 pt) Show that if \( f(x) \in R[[x]] \) is such that \( f(0) = \prod_{i=1}^{n} p_i^{a_i} \) (with the \( p_i \)'s distinct nonzero prime elements of \( R \) and each \( a_i > 0 \)) and \( f(x) = \prod_{j=1}^{t} f_j(x) \) (with each \( f_j(x) \) irreducible) then \( 1 \leq t \leq \sum_{i=1}^{n} a_i \). Give examples to show that both bounds can be achieved.
   c) (5 pt) Suppose that \( R \) is a PID. Show that if \( f(x) \neq x \) is irreducible in \( R[[x]] \) then \( f(x) = p^n + xg(x) \) with \( p \) a nonzero prime in \( R \) and \( g(x) \in R[[x]] \) (is the converse true?).
   d) (5 pt) With the notation as above, show that if \( R \) is a PID, then \( n \leq t \leq \sum_{i=0}^{n} a_i \).
5. Let $R$ be a domain with quotient field $K$. $\omega \in K$ is called almost integral over $R$ if there is a nonzero $r \in R$ such that $rx^n \in R$ for all $n \geq 0$. If $R$ contains all of the elements $\omega \in K$ that are almost integral over $R$, we say that $R$ is completely integrally closed.

a) (5 pt) Show that any UFD is completely integrally closed.

b) (5 pt) Suppose that $A \subseteq B$ are integral domains. Completely characterize when the domain $A + xB[x]$ is a UFD.