MATH 724
FALL 2005
HOMEWORK 3

Due Wednesday October 19, 2005.

1. (5 pt) Show that R is a PID if and only if R is a UFD of dimension no more than 1.

2. (5 pt) Let R be a commutative ring with identity and T an extension of R that is (finitely) generated by m elements over R. If the Krull dimension of R is n, show that

$$\dim(T) \leq 2^m n + 2^m - 1.$$

What is the lower bound for $\dim(T)$?

3. Recall that V is a valuation domain if given two nonzero elements $a, b \in V$ then either a divides b or b divides a. A Bezout domain is a domain where every finitely generated ideal is principal and a GCD domain is a domain where every two (nonzero) elements have a greatest common divisor.

a) (5 pt) Show that any we have the implications

$$\text{Valuation domain} \Longrightarrow \text{Bezout domain} \Longrightarrow \text{GCD domain}.$$

b) (5 pt) Show that none of the implications above are reversible.

c) (5 pt) Show that R is a valuation domain if and only if it is Bezout and quasi-local.

d) (5 pt) Show that R is a Bezout domain if and only if it is a GCD domain with the property that $\gcd(a, b)$ is a linear combination of a and b.

e) (5 pt) Show that any GCD domain (and hence Bezout and valuation) is integrally closed.