MATH 728
FALL 2004
HOMEWORK 1

Due Wednesday September 1, 2004.

1. (5 pt) Give an example of a division ring that is *not* a field.

2. Consider the set $\mathfrak{C} = \{f | f : \mathbb{R} \longrightarrow \mathbb{R}\}$ of functions from $\mathbb{R}$ to $\mathbb{R}$. Also consider the collection of functions $X = \{f_\alpha | \alpha \in \mathbb{R}\}$ where $f_\alpha(x)$ is defined by

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x = \alpha, \\ 0 & \text{if } x \neq \alpha. \end{cases}$$

   a) (5 pt) Show that $\mathfrak{C}$ is a real vector space.
   b) (5 pt) Show that X is a linearly independent subset of $\mathfrak{C}$.
   c) (5 pt) If X a basis for $\mathfrak{C}$ over $\mathbb{R}$? If not can you construct a basis for $\mathfrak{C}$ over $\mathbb{R}$ containing X?

3. Let $\mathbb{K} \subseteq \mathbb{F}$ be fields.
   a) (5 pt) Show that $\mathbb{F}$ is a vector space over $\mathbb{K}$.
   b) (5 pt) Show that if $\dim_{\mathbb{K}}\mathbb{F} = n < \infty$ then every element of $\mathbb{F}$ is algebraic over $\mathbb{K}$.
   c) (5 pt) Give an example of an infinite extension of fields that is algebraic and an example of an infinite extension of fields that is not algebraic.