MATH 728
FALL 2004
HOMEWORK 2

Due Wednesday September 8, 2004.

1. Let $W \subseteq V$ be vector spaces over $\mathbb{F}$.
   a) (5 pt) Show that $\dim(W) \leq \dim(V)$.
   b) (5 pt) Show that $\dim(V) = \dim(W) + \dim(V/W)$.

2. (5 pt) Let V and W be subspaces of some vector space U. Show that

$$\dim(V) + \dim(W) = \dim(V \bigcap W) + \dim(V + W).$$

What happens in the case where $U = V \oplus W$?

3. *In this problem we explore the notion of a "dual space" which will reappear in many forms.*

Let V be a vector space over $\mathbb{F}$. We define $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$
   a) (5 pt) Show that V^* is a vector space over $\mathbb{F}$.
   b) (5 pt) Show that if V is finite dimensional, then $V^* \cong V$.
   c) (5 pt) Show that if V is finite dimensional then there is an isomorphism $V \cong V^{**}$ that is independent of the choice of basis of V.
   d) (5 pt) Show that if V is infinite dimensional then $V^* \not\cong V$.