MATH 728
FALL 2004
HOMEWORK 3

Due Friday September 17, 2004.

1. Let V be an inner product space over $\mathbb{R}$. Suppose that $\{e_i\}_{i\in I}$ is an orthonormal basis for V and consider the dual map $\phi_i : V \longrightarrow \mathbb{R}$ given by

$$\phi_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Finally, let $W \subseteq V^*$ be the subspace of $V*$ spanned by $\{\phi_i\}_{i\in I}$.

a) (5 pt) Show that if $\phi \in W$ then there is a unique $u \in V$ such that $\phi(v) = \langle u, v\rangle$ for all $v \in V$.

b) (5 pt) Conclude that if V is finite dimensional, then every linear functional on V (that is, a linear transformation from V to $\mathbb{R}$) is of the form $\langle \circ, u\rangle$ for some (unique) $u \in V$.

2. (5 pt) Consider the real vector space $V = \oplus_{i\in I}\mathbb{R}$. Show that the standard "dot product" extends to an inner product on this space.

3. (5 pt) Show that if V is a real inner product space, then $\|v\| = \sqrt{\langle v, v\rangle}$ is a norm on V.

4. (5 pt) Let $\mathfrak{C}$ be the vector space of continuous functions from $[0, 1]$ to $\mathbb{R}$. Show that $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$ defines an inner product on $\mathfrak{C}$.

5. Consider the set of functions $\sin(n\pi x)$, $n \geq 1$ on the interval $[0, 1]$. Use the inner product given in problem 4.

a) (5 pt) Show that this set of functions is orthogonal.

b) (5 pt) Adjust the set so that it is an orthonormal set.

c) (5 pt) Consider the continous function $f(x) = 2x$. For each $n \geq 1$ compute $\langle 2x, \sin(n\pi x)\rangle$ (you have computed the Fourier sine coefficients of the function $f(x) = 2x$).

d) (5 pt) Show that the set of continuous functions on $[0, 1]$ is infinite dimensional.