MATH 728
FALL 2004
HOMEWORK 4

Due Monday September 27, 2004.

1. (5 pt) Suppose that M and N are $R-$modules and that there are one to one $R-$module homomorphisms $f : M \longrightarrow N$ and $g : N \longrightarrow M$ (so one can identify M as a submodule of N and N as a submodule of M). Does it follow that $M \cong N$? Prove this or give a counterexample.

2. (5 pt) Let M and N be $R-$modules. Suppose that you have $R-$module homomorphisms $f : M \longrightarrow N$ and $g : N \longrightarrow M$ such that $fg = 1_N$ and $gf = 1_M$. Show that $N = \text{im}(f) \oplus \ker(g)$ (and hence $M = \text{im}(g) \oplus \ker(f)$).

3. (5 pt) Consider the following commutative diagram of $R-$module homomorphisms

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Show that if the columns and the top two rows are exact, then the bottom row is exact.

4. (5 pt) Show that if R is commutative with identity and M is an $R-$module then there is an $R-$module isomorphism

$$\text{Hom}_R(R, M) \cong M.$$

5. (5 pt) Let R be commutative with identity. Show that every $R-$module is free if and only if R is a field.