MATH 728
FALL 2004
HOMEWORK 5

Due Wednesday October 6, 2004.

A category, $\mathfrak{C}$, is a collection of objects together with the following.

a) A collection of disjoint sets, one for each pair of objects $A, B \in \mathfrak{C}$, denoted $\hom(A, B)$. An element $f \in \hom(A, B)$ is called a morphism from A to B and is sometimes written $f : A \longrightarrow B$.

b) For each triple (A, B, C) of objects in $\mathfrak{C}$, we have a function

$$\hom(B, C) \times \hom(A, B) \longrightarrow \hom(A, C)$$

(if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we write $(g, f) \mapsto g \circ f$). This is called the composite and is subject to the following two axioms:

i) $h \circ (g \circ f) = (h \circ g) \circ f$.

ii) For any object A, there is a morphism $1_A : A \longrightarrow A$ such that for all $f : B \longrightarrow A$ and $g : A \longrightarrow B$, $g \circ 1_A = g$ and $1_A \circ f = f$.

A functor from the category $\mathfrak{C}$ to the category $\mathfrak{D}$ is a pair of functions (both denoted by F) such that $F(C)$ is an object of $\mathfrak{D}$ for all objects $C \in \mathfrak{C}$. Also if $f : A \longrightarrow B$ is a morphism, then $F(f) : F(A) \longrightarrow F(B)$ is a morphism with the following conditions.

a) $F(1_A) = 1_{F(A)}$ for all objects A in $\mathfrak{C}$.

b) $F(g \circ f) = F(g) \circ F(f)$ (in this case the functor is called covariant). OR

b$'$) $F(g \circ f) = F(f) \circ F(g)$ (in this case the functor is called contravariant).

1. Show that the following form categories.

a) (5 pt) Commutative rings with identity (with ring homomorphisms).

b) (5 pt) Abelian groups (with group homomorphisms).

c) (5 pt) $R-$modules (with $R-$module homomorphisms).

2. (5 pt) Let $\mathfrak{C}$ be the category of commutative rings with identity and let $U(R)$ denote the units of R. Show that the assignment $R \mapsto U(R)$ defines a functor from the catogory of commutative rings with identity to the category of abelian groups (how does this work on morphisms?).

3. Consider a fixed $R-$module D. We have seen that for all $R-$modules A, $\mathrm{Hom}_R(D, A)$ is again an $R-$module.

a) (5 pt) Show that the assignment $A \mapsto \mathrm{Hom}_R(D, A)$ defines a covariant functor from the category of $R-$modules to itself (how does it work on morphisms?).

b) (5 pt) Show that the assignment $A \mapsto \mathrm{Hom}_R(A, D)$ defines a contravariant functor from the category of $R-$modules to itself (how does it work on morphisms?).