MATH 728
FALL 2004
FINAL EXAM

Due Wednesday December 15, 2004.

1. Suppose that A is a matrix (over a field, $\mathbb{F}$) with characteristic polynomial $f(x) = x^6 + 2x^4 - 7x^2 + 4$.

   a) (5 pt) Suppose that the equation $y^2 + 1 = 0$ has a solution in $\mathbb{F}$. Find all possible rational canonical forms, primary rational canonical forms, and Jordan forms (if possible).

   b) (5 pt) Suppose that the equation $y^2 + 1 = 0$ has no solution in $\mathbb{F}$. Find all possible rational canonical forms, primary rational canonical forms, and Jordan forms (if possible).

2. (5 pt) Show that if M is a nilpotent matrix over a field $\mathbb{F}$, then all the eigenvalues of M are 0. Use this to find all possible Jordan forms of a 5×5 matrix over a field $\mathbb{F}$.

3. (*The Picard group*) Let R be an integral domain with quotient field K. Let $\mathrm{Pic}(R)$ denote the isomorphism classes of rank 1 (finitely-generated) projective $R-$modules. For two isomorphism classes, $[P]$ and $[Q]$, we define multiplication via

$$[P] \circ [Q] = [P \otimes_R Q].$$

   a) (5 pt) Show that $\mathrm{Pic}(R)$ forms a monoid with this multiplication.

   b) (5 pt) If P is a finitely-generated projective $R-$module, show that $P^* = \mathrm{Hom}_R(P, R)$ is also a (finitely-generated) projective $R-$module.

   c) (5 pt) Show that if P is a (finitely-generated) rank 1 projective $R-$module, then we can identify P as a fractional invertible ideal of R (hint: use the fact that P is rank 1, tensor the injection $R \longrightarrow K$ with P; you may use the fact from class that a fractional ideal is invertible if and only if it is projective).

   d) (5 pt) Conclude that $\mathrm{Pic}(R)$ forms a group by showing that $[P]^{-1} = \mathrm{Hom}_R(P, R)$.

4. (5 pt) Let R be commutative with identity. Show that $K_0(R)$ is a direct summand of $K_0(R[x])$.

5. (5 pt) Let R be a Euclidean domain. Show that if M is an $n \times n$ matrix over R, then R can be reduced to a matrix of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

with $\lambda_1 | \lambda_2 | \cdots | \lambda_n$ by using elementary row and column operations (note: be careful if the matrix is singular...what is the technical definition of $a|b$?).