1. (5 pt) Let \( p \) be a prime integer. Characterize (and find the number of) units in the ring \( \mathbb{Z}/p^n\mathbb{Z} \). Use this to find a formula for the number of units in \( \mathbb{Z}/n\mathbb{Z} \) where \( n > 2 \).

2. Find all primes \( p \) such that:
   a) (5 pt) \(-7\) is a square mod(\( p \)).
   b) (5 pt) \( \frac{3}{5} \) is a square mod(\( p \)).

3. (5 pt) Show that if \( p \) is an odd prime then the units of \( \mathbb{Z}/p^n\mathbb{Z} \) form a cyclic group. What happens in the case that \( p = 2 \) (what is the group structure of \( U(\mathbb{Z}/2^n\mathbb{Z}) \))? 

4. Let \( d \) be a square-free integer (that is, \( d \) is divisible by no square except 1) and consider the ring \( R := \mathbb{Z}[\omega] \) where \( \omega \) is given by

\[
\omega = \begin{cases} 
\sqrt{d} & \text{if } d \equiv 2, 3 \pmod{4} \\
\frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}
\end{cases}
\]

These rings are called the quadratic rings of integers. If \( d > 0 \) the quadratic ring of integers is called real and if \( d < 0 \) then the quadratic ring of integers is called imaginary.

We define the norm \( (N) \) by \( N(a + b\omega) = (a + b\omega)(a + b\overline{\omega}) \) where

\[
\overline{\omega} = \begin{cases} 
-\sqrt{d} & \text{if } d \equiv 2, 3 \pmod{4} \\
\frac{1 - \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}
\end{cases}
\]

Verify the following properties of the norm.
   a) (5 pt) \( N(R) \subseteq \mathbb{Z} \).
   b) (5 pt) \( N(x) = 0 \) if and only if \( x = 0 \).
   c) (5 pt) \( N(xy) = N(x)N(y) \).
   d) (5 pt) \( x \in U(R) \) if and only if \( N(x) = \pm 1 \).

5. (5 pt) Consider the family of quadratic rings of integers defined above. Show that if \( R \) is an imaginary quadratic ring of integers, then \( U(R) = \pm 1 \) unless \( d = -1 \) or \( d = -3 \). What happens in these last two cases? By way of contrast, show that \( U(\mathbb{Z}[\sqrt{2}]) \) is infinite.