1. (5 pt) Show that the equation $x^4 + y^4 = z^2$ has no nontrivial solutions for $x, y, z \in \mathbb{N}$. Hint: show that given a solution with $x, y, z \in \mathbb{N}$, show that a “smaller” positive solution can be obtained. This technique is referred to as “infinite descent”. (Note that this problem takes care of the Fermat problem for $n = 4$.)

2. (5 pt) Let $R$ be an integral domain with quotient field $K$. We define the integral closure of $R$ to be

$$\overline{R} = \{\alpha \in K | p(\alpha) = 0 \text{ for some monic } p(x) \in R[x]\}.$$  
We say that $R$ is integrally closed if $R = \overline{R}$ (that is, $R$ already contains all of its integral elements from $K$). Prove that any UFD is integrally closed.

3. (5 pt) Let $R$ be an integral domain with quotient field $K$ and $p \in R$ a nonzero prime element. Show that $p$ is also a prime element of $R[x]$.

4. (5 pt) Let $F$ be a field extension of degree $n$ over $\mathbb{Q}$. Suppose that $\omega \in F$ is a root of a monic polynomial in $\mathbb{Z}[x]$. Show that $\omega$ is a root of a monic polynomial in $\mathbb{Z}[x]$ of degree no more than $n$ and, in particular, show that the minimal polynomial of $\omega$ (over $\mathbb{Q}$) may be taken to be monic and in $\mathbb{Z}[x]$.

5. (5 pt) Let $d$ be a square-free integer. Show that the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{d})$ is given by

$$R = \begin{cases} 
\mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \text{ mod}(4), \\
\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \text{ mod}(4).
\end{cases}$$