Notes. \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) are the integers, the rational numbers, the real numbers, and the complex numbers respectively. All rings have identity unless specifically indicated otherwise, and all \( R \)-modules are unitary.

1. Let \( p \in \mathbb{N} \) be prime. Show that any group of order \( p^2 \) is abelian.
2. Show that any group of order 280 is not simple.
3. Let a finite group \( G \) acts transitively on a finite set \( \Omega \) of cardinality greater than one. Show that there is an element of \( G \) that fixes no element of \( \Omega \).
4. Let \( R \) be a commutative ring with 1 and \( I \) an injective \( R \)-module. Show that if the sequence
   \[
   0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0
   \]
   is exact, then the sequence
   \[
   0 \rightarrow \text{Hom}_R(C, I) \xrightarrow{f^*} \text{Hom}_R(B, I) \xrightarrow{g^*} \text{Hom}_R(A, I) \rightarrow 0
   \]
   is also exact.
5. Let \( R \) be a commutative ring with 1 and let \( J \) be the intersection of all maximal ideals of \( R \).
   (a) Show that if \( x \in J \) and \( r \in R \) then \( 1 + rx \) is a unit in \( R \).
   (b) Show that if \( M \) is a finitely generated \( R \)-module with \( M = JM \) then \( M = 0 \).
6. Let \( M \) be a simple left \( R \)-module. Show that a homomorphism \( f: M \rightarrow M \) is either an isomorphism or the zero homomorphism and hence \( \text{End}_R(M) \) is a division ring.
7. Suppose \( I \) is a proper ideal of a domain \( R \) that is injective as a \( R \)-module, show \( I = 0 \).
8. Show that if \( R \) is an integral domain with the property that \( R/I \) is a finite ring for any nonzero ideal \( I \), then every nonzero prime ideal of \( R \) is maximal.
9. Find the minimal polynomial over \( \mathbb{Q} \) of the element \( \sqrt{2} + \sqrt{3} \in \overline{\mathbb{Q}} \) and find the Galois group of the Galois closure of \( \mathbb{Q}[\sqrt{2} + \sqrt{3}] \) over \( \mathbb{Q} \).
10. Show for a field \( K \) of characteristic \( p > 0 \) that the following are equivalent:
   (a) Every finite field extension of \( K \) is separable.
   (b) The Frobenius homomorphism \( F: K \rightarrow K \) given by \( F: x \mapsto x^p \) is an epimorphism.