Chapter 1

HALF-FACTORIAL DOMAINS, A SURVEY

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1. INTRODUCTION

Let D be an integral domain. D is *atomic* if every nonzero nonunit of D can be written as a product of irreducible elements (or atoms) of D. Let $\mathcal{I}(D)$ represent the set of irreducible elements of D. Traditionally, an atomic domain D is a unique factorization domain (UFD) if $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ for each α_i and $\beta_j \in \mathcal{I}(D)$ implies:

1. $n = m$,
2. there exists a permutation σ of $\{1, \ldots, n\}$ such that α_i and $\beta_{\sigma(i)}$ are associates.

Call an atomic domain D a *half-factorial domain (HFD)* if 1) holds. A well known result of Carlitz [13] shows that this definition has its roots in algebraic number theory. We restate and offer a proof of his result using the above definition.

Theorem 1 (Carlitz's Theorem [13]). *Let R be the ring of integers in a finite extension K of the rationals. R is an HFD if and only if K has class number 1 or 2.*

Proof. ($\Rightarrow$) Suppose $\mid Cl(R) \mid > 2$ where $Cl(R)$ represents the class group of R. Let $g \in Cl(R)$ with $n = \mid g \mid > 2$, P be a prime ideal of R of class g and Q be a prime ideal of class $-g$. Then $P^n = \alpha R, Q^n = \beta R$, and $PQ = \gamma R$ with α, β, and $\gamma \in \mathcal{I}(R)$. Now $\alpha\beta R = P^n Q^n = (PQ)^n = \gamma^n R$ implies that $\alpha\beta = u\gamma^n$ where u is some unit in R. Since $n > 2$, R is not an HFD. If every $g \in Cl(R)$ has order 2, then let g_1, g_2 and g_3 be elements of $Cl(R)$ with $g_1 \neq g_2$ and $g_3 = -(g_1 + g_2)$. Taking prime ideals P, Q and H from these classes (respectively) yields $P^2 = \alpha R, Q^2 = \beta R, H^2 = \gamma R$ and $PQH = \delta R$ where each of the generators above is irreducible in R. As in the first case, we get $\delta^2 = u\alpha\beta\gamma$ and again R is not an HFD.

($\Leftarrow$) If K has class number 1, then R is a UFD and we are done. Suppose K has class number 2 and that

$$\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m \tag{1.1}$$

where each α_i and $\beta_j \in \mathcal{I}(R)$. Without loss of generality, we can assume that none of these factors are primes. Hence the principal ideal generated by each of these irreducibles is the product of two nonprincipal primes. By counting the number of prime ideals on each side of the equation (1.1), $2n = 2m$ implies that $n = m$. $\square$

The reader should note that the proof of Theorem 1 is dependent on the fact that each nonzero ideal class of R contains a nonzero prime ideal. This is not true for a general Dedekind domain (see Proposition 9 in Section 3).

In this paper, we will review much of the recent literature concerning half-factorial domains. While our review is by no means exhaustive, our goal is to give the reader a solid introduction to this subject based on the major publications in this area starting with the papers of Zaks ([46] and [47]). We break the remainder of this summary into 4 sections. In Section 2 we review some basic facts and examples, including a proof that $\mathbb{Z}[\sqrt{-3}]$ is the unique non-integrally closed imaginary quadratic HFD. In Section 3 we consider the question of characterizing Krull and Dedekind domains which are HFDs. This leads to the study of "semi-length functions" and in particular the Zaks-Skula function. Such functions are instrumental for the analysis in this section of the case where the divisor class group (or class group) of the domain D is cyclic. Sections 4 and 5 deal with ring extensions. Section 4 develops a "boundary" condition which characterizes when an overring of an HFD is again an HFD. Section 5 gives a characterization of when a polynomial ring $R[X]$ is an HFD as well as some necessary conditions for a ring of the form $A + XB[X]$ to be an HFD.

The original idea for this article arose from an invited lecture given by the first author at the "Factorization in Integral Domains" mini-conference at the University of Iowa in March 1996 (see [1]). While that talk included some discussion of generalizations of the half-factorial property, we choose to not cover that topic here. A review of the congruence half-factorial and k-half-factorial properties can be found in a companion survey article in this volume [15]. Results concerning generalizations of the half-factorial property related to the study of overrings can be found in [8], [9] and [11].

2. EXAMPLES AND BASIC RESULTS

We begin with some basic examples demonstrating the half-factorial property.

Example 2. Since $D = \mathbb{Z}[\sqrt{-5}]$ has class number 2, D is an HFD but not a UFD. The *usual* specific factorization presented to show that unique factorization fails is

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

A complete argument that D is not a UFD must include verification that 2 (or 3) is not an associate of both $(1 + \sqrt{-5})$ and $(1 - \sqrt{-5})$.

Example 3 (Anderson-Anderson-Zafrullah). [3, Theorem 5.3] Here is perhaps the simplest construction of an HFD not involving an algebraic number ring. Let K be any field and $A \subseteq K$. If A is a field, then [3, Theorem 2.9] shows that the irreducible elements of $R = A + XK[X]$ are of the form

1. aX where $a \in K$, or
2. $a(1+Xf(X))$ where $a \in A$, $f(X) \in K[X]$ and $1+Xf(X)$ is irreducible in $K[X]$.

Thus, the number of elements in an irreducible factorization of a nonzero nonunit $g(X) \in R$ must be the same as the number of elements in a irreducible factorization of $g(X)$ in the UFD $K[X]$. It is then easy to argue that $R = A + XK[X]$ is an HFD if and only if A is a subfield of K. Hence $\mathbb{R} + X\mathbb{C}[X]$ and $\mathbb{Q} + X\mathbb{R}[X]$ are both HFDs. They are not UFDs since $X^2 = X \cdot X = (iX)(-iX)$ and $X^2 = X \cdot X = (\sqrt{2}X)(\frac{1}{\sqrt{2}}X)$ are respective nonunique factorizations in each domain.

Example 4. [2, Proposition 3.1] Example 3 is merely a special case of a stronger result obtained by the same authors using the $D+M$ construction. Let T be an integral domain of the form $K + M$, where M is a nonzero maximal ideal of T and K is a subfield of T. Let D be a subring of K and

$R = D + M$. Then R is an HFD if and only if D is a field and T is an HFD. Thus, if A is a subfield of K, then $R = A + XK[[X]]$ is also an HFD.

Example 5 (Zaks). [47] Unlike a UFD, an HFD need not be integrally closed. Let $R = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Since $\mid Cl(R) \mid= 1$, R is a PID and hence UFD. Set $R' = \mathbb{Z}[\sqrt{-3}]$. Now, since $-3 \equiv 1 \pmod 4$, R' is not integrally closed, but is an HFD. To show this, Zaks argues (using norms) that if r is irreducible in R', then r remains irreducible in R.

We will show in Theorem 7 a stronger result (namely that this example is unique among imaginary quadratic orders). In the meantime, we present a Theorem of Halter-Koch [36] which in some sense generalizes Zaks' argument of Example 5.

Theorem 6 (Halter-Koch). *[36] Let K be a quadratic number field with ring of integers O_K and A be an order in K with $f > 1$ its conductor. The following are equivalent:*

1. *A is an HFD.*
2. *O_K is an HFD, $O_K = A \cdot O_K^\times$ and f is either prime or twice an odd prime.*

By Theorem 6, for $2 \leq d < 100$, $\mathbb{Z}[\sqrt{d}]$ is an HFD and not a UFD if and only if $d =$ 5, 10, 12, 13, 15, 18, 21, 26, 29, 30, 34, 35, 39, 42, 44, 45, 50, 51, 53, 55, 58, 61, 66, 69, 70, 74, 76, 77, 78, 84, 85, 87, 91, 93, and 95.

To give a flavor for the applications of norms to rings of algebraic integers, we expand on Example 5. This result can also be obtained from a careful application of Theorem 6.

Theorem 7. *[22] The ring $\mathbb{Z}[\sqrt{-3}]$ is the unique, non-integrally closed imaginary quadratic HFD.*

Proof. We shall defer to [47] for the fact that $\mathbb{Z}[\sqrt{-3}]$ is an HFD. In this proof, we will let $d < 0$ and consider two cases. The first case will be when $d \equiv 2$ or 3 mod(4), and the second case will be when $d \equiv$1 mod(4). In the first case, we have that an order R has the form $\mathbb{Z} + n\mathbb{Z}[\sqrt{d}]$, where n is the index. The norm form associated with this ring is

$$f(x, y) = x^2 - dn^2y^2.$$

If p is a prime dividing n, then we shall say $n = kp$ and consider the element $n\sqrt{d}$. The norm of this element is dk^2p^2 and we claim that this element is irreducible. To see this, note that the norm of any proper divisor of this is less than dn^2, and so the form of the norm tells us that $n\sqrt{d}$ must be divisible by a rational integer, but clearly it is not. So we have the factorization in R given by:

$$(n\sqrt{d})(-n\sqrt{d}) = (p)(p)(k)(k)(d).$$

In particular, since the left hand side is an irreducible factorization, we have that R is not an HFD unless k=1 and d=-1. So in this case, the only possible orders are the ones of prime index in the Gaussian integers. We will now examine this possibility in depth. Let R be of index p in $\mathbb{Z}[i]$. So R is of the form $\mathbb{Z} + pi\mathbb{Z}$. We note that in R, the element $p + pi$ is irreducible; indeed, any proper divisor must have norm 2, p, $2p$, or p^2, and checking all of the possibilities shows that $p + pi$ is irreducible. The norm of $p + pi$ is $2p^2$, so we have the following factorizations in R:

$$(p + pi)(p - pi) = (2)(p)(p)$$

and so again, R is not an HFD.

In the second case, we assume that $d \equiv$1 mod (4). Here, R takes the form $\mathbb{Z} + n\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ with n being the index. We shall write the norm form $g(x, y)$ in two equivalent ways:

$$g(x, y) = x^2 + nxy + n^2y^2\frac{(1-d)}{4} = (x + \frac{n}{2}y)^2 - \frac{dn^2}{4}y^2.$$

Letting $x = 0$ and $y = 1$ in the above equations, we obtain an element of norm $(\frac{1-d}{4})n^2$. This element is irreducible. To see this, we note that any proper divisor of this element has a norm necessarily dividing $(\frac{1-d}{4})n^2$. Therefore, we conclude that $|y|$ cannot be greater than or equal to 2. If $y = \pm 1$, then the norm of the element is given by

$$x^2 \pm nx + (\frac{1-d}{4})n^2$$

and for this norm to divide $(\frac{1-d}{4})n^2$, we necessarily must have

$$x^2 \pm nx \leq -(\frac{1-d}{8})n^2.$$

Some elementary calculus shows, however, that this implies that $d \geq$-1 which is a contradiction. Therefore, $y = 0$, and the divisor of the element $n(\frac{1+\sqrt{d}}{2})$ must be a rational integer, which is a contradiction.

We now conclude that we have the following factorizations in R:

$$(n(\frac{1+\sqrt{d}}{2}))(n(\frac{1-\sqrt{d}}{2})) = (\frac{1-d}{4})(n)(n).$$

Since the left hand side of the above equation is an irreducible factorization, we have contradicted HFD unless d=-3 and n is prime. Therefore, the only possible non-integrally closed HFDs are the ones of prime index in $\overline{\mathrm{R}}=\mathbb{Z}[\omega]$ where $\omega = \frac{1+\sqrt{-3}}{2}$ is the primitive complex sixth root of unity. Assume that the index of R in $\overline{\mathrm{R}}$ is a prime $p > 2$. Consider the element $p + p\omega \in$R. The norm of this element is $3p^2$, and the general norm polynomial is $h(x, y) = x^2 + pxy + p^2y^2$. So we see that this element is reducible only if there is an element of norm 3, but it is easy to check that none of the six elements of norm 3 in $\overline{\mathrm{R}}$ are in R. Therefore, we have the following factorizations in R:

$$(p + p\omega)(p + p\overline{\omega}) = (3)(p)(p)$$

where $\overline{\omega}$ is the conjugate of ω. As before, the left hand side of the above is an irreducible factorization, and so R is not an HFD. The only case that remains is the case of index 2 ($\mathbb{Z}[\sqrt{-3}]$) and this has been shown to be an HFD by Zaks [47]. $\square$

3. DEDEKIND AND KRULL EXAMPLES

In [42] and [43] Narkewicz poses the question of characterizing all Dedekind domains which satisfy the factorization property of Carlitz's Theorem (Theorem 1). Zaks and Skula both answered this question in similar manners for Dedekind domains with torsion class groups. If D is such a Dedekind domain and α is a nonprime irreducible of D, then

$$\alpha D = P_1 \cdots P_k$$

where $P_1, \ldots, P_k$ are nonprincipal prime ideals of D. If $[P_i]$ represents the divisor class of P_i in $Cl(D)$ and $\mid [P_i] \mid$ the order of $[P_i]$ in $Cl(D)$, then let

$$z(\alpha) = \sum_{i=1}^{k} \frac{1}{\mid [P_i] \mid}.$$

Setting $z(u) = 0$ when u is a unit of D induces a function

$$z : D^* \longrightarrow \mathbb{Q}$$

such that $z(\alpha\beta) = z(\alpha) + z(\beta)$. A function, z, with the properties mentioned above is called a *semilength function* on D (see [2] and [4]) and the particular function z above is referred to in the literature as the *Zaks-Skula function* [19]. For a given $\alpha \in D^*$, $z(\alpha)$ is also referred to as the *cross number of* α and

$$K(D) = \sup\{z(\alpha) \mid \alpha \in \mathcal{I}(D)\}$$

as the *cross number of* D. For more information on the cross number, the reader can consult [14], [27], [29], [30] and [40]. There is a close connection between the Zaks-Skula function, the cross number and the half-factorial property.

Theorem 8. *1) [47, Lemma 1.3] An atomic integral domain D is an HFD if and only if there is a semilength function on D with range $\mathbb{Z}^+$ such that $z(x) = 1$ for all irreducibles $x \in D$.*

2) [47, Theorem 3.3] [44, Theorem 3.1] Let D be a Dedekind domain with torsion class group. D is an HFD if and only if $z(\alpha) = 1$ for all irreducibles $\alpha \in D$.

Proof. We refer the interested reader to [47] for a proof of 1). We offer a proof of 2). ($\Leftarrow$) Suppose $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ for α_i and β_j in $\mathcal{I}(D)$. Then $z(\alpha_1) + \cdots + z(\alpha_n) = z(\beta_1) + \cdots + z(\beta_m)$ implies that $n = m$.

($\Rightarrow$) Let α be irreducible in D with $\alpha R = P_1 \cdots P_k$ for nonprincipal prime ideals $P_1, \ldots, P_k$ in D. Set $G = Cl(D)$, $t = \exp(G)$, $\mid [P_i] \mid = n_i$, and $n_i s_i = t$. Now $\alpha^t = P_1^{n_1 s_1} \cdots P_k^{n_k s_k} = (P_1^{n_1})^{s_1} \cdots (P_k^{n_k})^{s_k}$. Since each $P_i^{n_i}$ is principal generated by an irreducible we have that $s_1 + \cdots + s_k = t$. Thus $\frac{s_1 + \cdots + s_k}{t} = 1$ implies $\sum_{i=1}^k \frac{s_i}{t} = \sum_{i=1}^k \frac{1}{n_i} = 1$. $\square$

Part 2) of Theorem 8 depends solely on the distribution of prime ideals in the class group of the Dedekind domain D. In particular, it relies on certain types of finite sequences in $Cl(D)$. If G is a finite abelian group, then the sequence $T = \{g_1, \ldots, g_t\}$ is called a *block* if $\sum_{i=1}^t g_i = 0$. For simplicity, to represent blocks we use the notation $T = g_1 \cdots g_t$. T is an *irreducible block* if T contains no proper subblock. Let

$$\mathcal{B}(G) = \{T \mid T \text{ is a block of G}\}.$$

If $T_1 = g_1 \cdots g_r$ and $T_2 = h_1 \cdots h_v$ are blocks of G, then the operation $T_1 T_2 = g_1 \cdots g_r h_1 \cdots h_v$ makes $\mathcal{B}(G)$ an atomic monoid [37]. In more generality, if $\emptyset \neq G_0 \subseteq G$ then set

$$\mathcal{B}(G_0) = \{T \mid T = g_1 \cdots g_r \in \mathcal{B}(G) \text{ with } g_i \in G_0 \text{ for all } i\} \subseteq \mathcal{B}(G).$$

$\mathcal{B}(G_0)$ is an atomic submonoid of the monoid $\mathcal{B}(G)$.

The precise connection between the monoids $\mathcal{B}(G_0)$ and factorization properties of certain integral domains can be seen as follows. Let R be a Krull domain with divisor class group G and G_0 the set of divisor classes of G which contain height-one prime ideals of R. If R^* represents the nonzero elements of R and $\alpha \in R^*$, then

$$\alpha R = (P_1 \cdots P_k)_v$$

for unique height-one prime ideals $P_1, \ldots, P_k$ of R. The map

$$f : R^* \longrightarrow \mathcal{B}(G_0)$$

defined by

$$f(\alpha) = ([P_1] \cdots [P_k])$$

is a length preserving monoid homomorphism (see Geroldinger [26, Proposition 1]). Hence, factorization problems dealing with lengths of irreducible factorizations in R can be viewed as identical problems on the atomic monoid $\mathcal{B}(G_0)$. In particular, R is an HFD if and only if $\mathcal{B}(G_0)$ is half-factorial as a monoid. Thus, it is of interest to characterize Krull (or more specifically Dedekind) domains according to the distribution of prime ideals in their class group. One such characterization for Dedekind domains is offered in [31].

Proposition 9. *[31, Theorem 5] Let G be a countably generated abelian group and $\emptyset \neq S \subseteq G$. There exists a Dedekind domain D with class group isomorphic to G such that the classes that contain maximal ideals are precisely the elements of S if and only if S generates G as a monoid.*

Given an abelian group G and nonempty subset $S \subseteq G$ which satisfies the hypothesis of Proposition 9, the set $\{G, S\}$ is called a *realizable pair*. There is a simple form of the characterization above for Dedekind domains when G is a torsion group [35, Corollary 1.5]: $\{G, S\}$ is a realizable pair if and only if S generates G as a group. If $G = \mathbb{Z}$, then Proposition 9 can be restated as follows: $\{G, S\}$ is a realizable pair if and only if S generates $\mathbb{Z}$ as a group and contains both positive and negative elements.

Example 10. Let $G = \sum_{i=1}^{k} \mathbb{Z}_{n_i}$ and

$$S = \{e_1, \ldots, e_k\}$$

where the e_i are the standard basis vectors for G. By Proposition 9, there is a Dedekind domain D with realizable pair $\{G, S\}$. By Geroldinger's result [26, Proposition 1], we need only examine the monoid $\mathcal{B}(S)$ to determine if D is an HFD. In $\mathcal{B}(S)$ the only irreducible blocks are $e_1^{n_1}, \ldots, e_k^{n_k}$ which all have Zaks-Skula constant 1. Hence any Dedekind domain associated to $\{G, S\}$ is an HFD.

Example 10 implies that any finite abelian group G can serve as the class group of a Dedekind HFD. Thus, Carlitz's Theorem (Theorem 1) fails for general Dedekind domains. The construction technique used in Example 10 was extended by Zaks to show the following.

Theorem 11 (Zaks). *[46, Theorem 3] Let G be a finitely generated abelian group. Then there exists a Dedekind domain D with class group G such that D is an HFD.*

Proof. Write

$$G = \sum_{i=1}^{t} \mathbb{Z}_{n_i} \oplus \sum_{j=1}^{s} \mathbb{Z}$$

where s and t are nonnegative integers and n_i is a positive integer with $n_i \mid n_{i+1}$ for $1 \le i \le t-1$. Set

$$S = \{e_1, \ldots, e_t, e_{t+1}, \ldots, e_{t+s}, -e_{t+1}, \ldots, -e_{t+s}\}.$$

S generates G as a monoid and by Proposition 9, there is a Dedekind domain D with realizable pair $\{G, S\}$. Since G may not be torsion, we cannot use part 2) of Theorem 8 to argue that D is an HFD. Instead, we note that if x is a nonprime irreducible of D, then the ideal (x) is of the form $\prod_{k=1}^{n_i} P_k$ where each P_k is a prime ideal taken from the class e_i (for some fixed $1 \le i \le t$) or PQ where P and Q are prime ideals taken respectively from the classes e_j and $-e_j$ (for some fixed $t+1 \le j \le t+s$). If $y = \alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ are two different factorizations of an element $y \in D$ into irreducibles, then $n = m$ follows by counting the number of prime ideals of each class in the prime factorization of the ideal (y). □

Michael and Steffan [41] have further extended the previous result of Zaks as follows.

Theorem 12. *[41, Corollaire 6.1 and Proposition 8] Let G be an abelian group which is either*

1) *free,*
2) *torsion with finite exponent, or*
3) *divisible.*

Then there exists a Dedekind HFD with class group G.

The question of whether or not Theorem 12 holds for all abelian groups G is still open.

Given an abelian group G, exactly which realizable pairs $\{G, S\}$ have associated Dedekind domains which are HFDs? This is a question which has attracted attention in the papers [14], [17], [18], [25] and [26]. We list some basic results concerning this problem when the class group G is cyclic. In the finite case, we will list elements of $\mathbb{Z}_n$ in the form $\bar{i} = i + \mathbb{Z}$ and assume that $0 \le i \le n-1$.

Proposition 13. *Let $G = \mathbb{Z}_n$ for $n > 2$ and suppose $\{G, S\}$ is a realizable pair with associated Dedekind domain D.*

1) *If* $S = \{\bar{i}\}$*, then* D *is an HFD.*

2) *[15, Lemma 25] If* $\overline{1} \in S$ *and* D *is an HFD then* $r \mid n$ *for all* $\overline{r} \in S$*.*

3) *[17, Theorem 3.8] Let* $S = \{\overline{1}, \overline{r_1}, \overline{r_2}\}$*. Then* D *is an HFD if and only if* $r_i \mid n$ *for* $1 \leq i \leq 2$*.*

4) *[14, Theorem 3.10] If* $S = \{\overline{r_1}, \overline{r_2}, \overline{r_3}\}$ *and* $r_i \mid n$ *for* $1 \leq i \leq 3$*, then* D *is an HFD.*

5) *If* $n = p$ *is prime, then* D *is an HFD if and only if* $S = \{\bar{i}\}$*.*

6) *[17, Theorem 3.11] If* $n = p^k$ *for* $k \geq 2$ *and* $\overline{1} \in S$*, then* D *is an HFD if and only if* $S \subseteq \{\overline{1}, \overline{p}, \overline{p^2}, \dots, \overline{p^{k-1}}\}$*.*

Proof. Using a simple automorphism argument, the proof of 1) follows from Example 10 and the proof of 5) follows from 2). The remaining proofs can be found as listed above. Note that the proof cited above for 4) is dependent on a property of splittable sets discussed in [24]. □

Example 14. Part 3) of Proposition 13 cannot be improved. If $n = 30$ and $S = \{\overline{1}, \overline{6}, \overline{10}, \overline{15}\}$ then $T = \overline{1} \cdot \overline{6} \cdot \overline{6} \cdot \overline{6} \cdot \overline{6} \cdot \overline{10} \cdot \overline{10} \cdot \overline{15}$ is an irreducible block with

$$z(T) = \frac{1}{30} + 4 \cdot \frac{1}{5} + 2 \cdot \frac{1}{3} + \frac{1}{2} = 2.$$

Hence any Dedekind domain associated to $\{\mathbb{Z}_{30}, S\}$ is not an HFD.

For the case where the class group is infinite cyclic, much less is known. This case has been studied in detail in the papers [18], [6] and [7]. We begin with a fundamental fact in the class group $\mathbb{Z}$ case.

Proposition 15. *[6, Theorem 2.4] Let* D *be a Dedekind domain with realizable pair* $\{\mathbb{Z}, S\}$*. If* D *is an HFD, then there exists an integer* N *such that either*

1) $s_i < N$ *for all* $s_i \in S$*, or*

2) $N < s_i$ *for all* $s_i \in S$*.*

We shall refer to S as being *bounded above* (case 1)) or *bounded below* (case 2)). By using the automorphism of $\mathbb{Z}$ which sends 1 to -1, we can reduce the problem to one of considering only sets S which are bounded below. Hence, suppose

$$S = \{-m_1, \dots, -m_t, n_1, n_2, \dots\} \tag{1.2}$$

where the m_i and n_j are all positive integers and that $p_1, p_2, \ldots, p_k$ is a list of distinct prime integers such that

$$m_1 = p_1^{x_{11}} p_2^{x_{12}} \cdots p_k^{x_{1k}}, \ldots, m_t = p_1^{x_{t1}} p_2^{x_{t2}} \cdots p_k^{x_{tk}}$$

where the x_{ij} are nonnegative integers. Set

$$\mathcal{J} = \{i \mid \text{there exists } j \text{ and } k \text{ such that } x_{ji} \neq x_{ki}\}$$

and

$$\ll m_1, \ldots, m_t \gg = \prod_{i \in \mathcal{J}} p_i^{\max\{x_{1i}, x_{2i}, \ldots, x_{ti}\}}.$$

Call S *c-divisible* if for each i there is a positive integer d_i such that $n_i = d_i \cdot c$.

Proposition 16. *Let D be a Dedekind domain with realizable pair $\{\mathbb{Z}, S\}$ where S is of the form (1.2).*

1) *[6, Corollary 3.3(1)] If $t = 1$ and $m_1 = 2$ then D is an HFD.*
2) *[18, Corollary 4.4] If $\mid S \mid = 2$ then D is an HFD.*
3) *[7, Corollary 3] If $t \geq 2$ and D is an HFD, then S is $\ll m_1, \ldots, m_t \gg$-divisible.*
4) *[7, Theorem 8] If $t = 2$ and $\gcd(m_1, m_2) = 1$, then D is an HFD if and only if S is $\ll m_1, m_2 \gg$-divisible.*

The converse of part 3) of Proposition 16 is false. The interested reader is directed to Example 4 of [7] for a counterexample. A complete characterization of Dedekind domains with class group $\mathbb{Z}$ which are HFD is not known. While not directly related to the half-factorial property, readers with further interest in factorization properties of Krull domains with infinite cyclic divisor class group are directed to an amazing result in a recent paper of Kainrath [38]. Let D be such a Krull domain such that each divisor class of $Cl(D) = \mathbb{Z}$ contains a height-one prime ideal. The main result of [38] implies that if M is any nonempty finite subset of $\mathbb{N} - \{1\}$, then M is the set of lengths of irreducible factorizations of some nonzero nonunit in D.

4. ON INTEGRAL EXTENSIONS

The next two sections of this paper will highlight some important results concerning the behavior of ring extensions of HFDs. As HFDs are a natural generalization of UFDs, it is only fitting that we compare and contrast their respective ring-theoretic properties. For example, if R is a UFD, then it must be integrally closed. This property is not shared by HFDs in general (see Example 5). It is natural, therefore, to ask if the integral closure of an HFD is an HFD (this question was originally posed to the first author by V. Barucci). In this section, we shall examine the known results in this vein.

We open by noting that while it is well known that any localization of a UFD is again a UFD, the corresponding result does not hold for HFDs. We demonstrate this by example.

Example 17. Let D be a Dedekind domain with realizable pair $\{\mathbb{Z}_6, S_D\}$ where $S_D = \{\bar{1}, \bar{2}, \bar{3}\}$. By Proposition 13 part 3), D is an HFD. Set

$$\mathcal{Q} = \{Q \mid Q \text{ is a prime ideal of } D \text{ with } [Q] = \bar{1} \text{ or } [Q] = \bar{2}\}.$$

Now, suppose that P is a prime ideal with $[P] = \bar{3}$ and that $P \subseteq \cup_{Q\in\mathcal{Q}} Q$. By the main theorem of [45], $P = Q$ for some $Q \in \mathcal{Q}$, a contradiction. Hence $P \nsubseteq \cup_{Q\in\mathcal{Q}} Q$. Pick $t \in P \setminus \cup_{Q\in\mathcal{Q}} Q$ and set $T = \{1, t, t^2, \dots\}$. If $R = D_T$, then R is a Dedekind domain with realizable pair $\{Cl(R), S_R\}$ and (see [8, Theorem 2])

1. $Cl(R) \cong Cl(D)/(\ker\tau)$ where τ is the natural map from $Cl(D) \longrightarrow Cl(R)$ defined by $\tau : [I] \longrightarrow [IR]$ and
2. $S_R = \tau(S_D)\setminus\{0\}$.

Thus, $S_R = \{\bar{1}, \bar{2}\}$ and $Cl(R) \cong \mathbb{Z}_6/\mathbb{Z}_2 \cong \mathbb{Z}_3$. By Proposition 13 part 2), R is not an HFD.

In this section, unless otherwise stated, R will denote an HFD and we will denote the integral closure of R by $\overline{R}$. To facilitate our study of this problem, we present the *boundary map*, which is a simple generalization of the length function introduced by Zaks ([46], [47]).

Definition 1. *[21] Let R be an HFD with quotient field K. If $R \neq K$, we define $\partial_R : K \setminus 0 \longrightarrow \mathbb{Z}$ by $\partial_R(\alpha) = n - m$ where $\alpha = \frac{\pi_1\pi_2\cdots\pi_n}{\xi_1\xi_2\cdots\xi_m}$ where π_i, ξ_j are irreducible elements of R for $1 \leq i \leq n$ and $1 \leq j \leq m$. If $R = K$ we say that $\partial_R(\alpha) = 0$ for all $\alpha \in R$.*

We remark at this point that ∂_R is a well-defined surjection onto the rational integers precisely because of the fact that R is an HFD. It is also worth noting that the restriction of ∂_R to the HFD R is compatible with Zaks' length function (see [47]).

The boundary map behaves quite well in conjunction with (almost) integral elements in the following sense.

Proposition 18. *[21, Lemma 2.3] Let R be an HFD with quotient field K and let $\alpha \in K$ be almost integral over R. Then $\partial_R(\alpha) \geq 0$*

Proof. As α is almost integral over R, there exists an $r \in R$ such that $r\alpha^n \in R$ for all $n > 0$. Using the properties of ∂_R, we obtain

$$\partial_R(r\alpha^n) = \partial_R(r) + n\partial_R(\alpha) \geq 0.$$

As the above inequality holds for all nonnegative integers n, we have that $\partial_R(\alpha) \geq 0$. $\square$

Intuitively, this result means that (almost) integral elements of the quotient field cannot be formed with more factors in the denominator than in the numerator. However, the techniques used do not provide an obstruction to the possibility of nonunits of boundary 0 (i.e. an equal number of factors in the top and bottom of the fraction). This will present some difficulties that will become clearer soon.

The above techniques do, however lead to the following result. We note that in this result, integrality is not used.

Theorem 19. *[21, Theorem 2.5],[23] Let R be an HFD and let S be an overring of R such that no nonunit of S has boundary 0. Then S is an HFD if and only if $\partial_R(\alpha) = 1$ for all irreducible elements of S.*

Proof. We ignore the case where S is a field as the result holds trivially. The key to this proof is the fact that $\partial_R(\alpha) \geq 0$ for all $\alpha \in S$. Indeed, if there exists $\alpha \in S$ with $\partial_R(\alpha) = n < 0$, then we choose r to be an irreducible element of R (such that r is a nonunit in S) and note that the element αr^{-n} is a nonunit of S with boundary 0. This establishes our first claim. (It is of note that a similar application of the above technique shows that any unit in S must have boundary 0).

With the above fact in hand, we observe that no nonunit of R becomes a unit in S, for if r is a nonunit in R that becomes a unit in S, then $r^{-1} \in S$ has negative boundary. We also note that under these hypotheses, every irreducible element of R remains irreducible in S. Indeed, any irreducible element $r \in R$ has $\partial_R(r) = 1$. Since r cannot be a unit in S and there are no nonunits of S with boundary 0, then r must be irreducible in S.

To complete the proof of the above theorem, we assume that we can find an irreducible element $\alpha \in S$ such that $\partial_R(\alpha) = n > 1$. We write

$$\alpha = \frac{\pi_1\pi_2\cdots\pi_{k+n}}{\xi_1\xi_2\cdots\xi_k}$$

where the elements π_i, ξ_j are irreducible elements of R (and hence irreducible in S). Multiplying both sides of the above equation by the denominator of the right hand side, we obtain

$$\xi_1\xi_2\cdots\xi_k\alpha = \pi_1\pi_2\cdots\pi_{n+k}.$$

As $n > 1$ and α is irreducible, we have that S is not an HFD.

For the other direction, we first note that S is necessarily atomic. Indeed if $\alpha \in S$, then $\partial_R(\alpha) = n \geq 0$ and this n gives an upper bound on the number of factors that a given factorization of α could possess (since there are no nonunits of boundary 0). Now assume that we have the following irreducible factorizations in S:

$$\xi_1\xi_2\cdots\xi_m = \pi_1\pi_2\cdots\pi_n.$$

Applying the boundary to both sides of the above, and recalling that the boundary of any irreducible in S is 1, we obtain $m = n$ and hence S is an HFD.

□

The central ideas of the above results revolve around the nonexistence of nonunits of S with 0 boundary. The additional assumption of integrality does not seem to circumvent this potential hazard, so we would conjecture that more a more appropriate topic to investigate would be HFDs in overrings of this type.

Question 20. Let S be an overring of R, an HFD. If S possesses no nonunit of boundary 0 then S is atomic. Is the converse true?

We would also conjecture that it is possible for the integral closure of an HFD to be a non-HFD via the loss of atomicity (see [23]). This would lead to the following question.

Question 21. Let R be an HFD. If $\overline{R}$ is atomic, then is $\overline{R}$ an HFD?

It would also be interesting to know what happens when we replace the assumption "atomic" with "Noetherian" and when we replace $\overline{R}$ with S, an overring containing no nonunits of boundary 0.

A recent application of these techniques has given a partial answer to the question on the behavior of the integral closure of an HFD.

Theorem 22. *[21, Theorem 3.1] Let $F/\mathbb{Q}$ be an algebraic number field with ring of integers $\overline{R}$. If $R \subseteq \overline{R}$ is an order with the HFD property, then $\overline{R}$ is an HFD.*

Although we omit the proof here, we remark that what makes this work is the fact that every irreducible in $\overline{R}$ can be thought of as a irreducible in

R (up to a unit in $\overline{R}$). This is a recurring theme from this section and from others (c.f. Halter-Koch [33]).

The above result coupled with Theorem 7 motivates the following questions along the line of Gauss' conjecture (on the infinitude of real quadratic UFDs).

Question 23. Are there an infinite number of (integrally closed) real quadratic HFDs?

Question 24. Does there exist a real quadratic HFD containing infinitely many orders that also have the half-factorial property?

5. ON POLYNOMIAL AND POLYNOMIAL-LIKE EXTENSIONS

We continue our view toward the interplay of ring theoretic properties possessed by UFDs and HFDs. A standard (and very important) result from algebra states that if R is a UFD then so is the polynomial ring $R[x]$. From this it follows that if R is a UFD, then so is $R[X]$ where X denotes any family of indeterminates.

It is natural to ask to what extent these results extend to HFDs, and although the theory in the half-factorial context is not as sweeping, it is certainly more complete than the known results for integral extensions.

In this section, we will give some results that actually give a complete classification of Noetherian polynomial HFDs. In the non-Noetherian case there are still open questions, but a necessary condition will be shown for $R[x]$ to be an HFD.

In lieu of proving the main theorem shown in this section, we will look at an example that, although somewhat simple, contains all the key ingredients that go in to the proof. The motivating question for our example will be "is the ring $\mathbb{Z}[\sqrt{-3}][x]$ an HFD?" Recall that the ring $\mathbb{Z}[\sqrt{-3}]$ is an HFD (otherwise the answer to our motivating question would be a resounding "NO" right from the start).

We note that the integral closure of $\mathbb{Z}[\sqrt{-3}]$ is $\mathbb{Z}[\omega]$ where $\omega = \frac{-1+\sqrt{-3}}{2}$ denotes a primitive third root of unity. Noting that the irreducible polynomial of ω over $\mathbb{Z}[\sqrt{-3}]$ is x^2+x+1, we consider the following factorizations in the polynomial ring $\mathbb{Z}[\sqrt{-3}][x]$

$$(2x + (1 + \sqrt{-3}))(2x + (1 - \sqrt{-3})) = (2)(2)(x^2 + x + 1).$$

It is an easy check to see that every factor in the above expression is irreducible in $\mathbb{Z}[\sqrt{-3}][x]$, so the ring $\mathbb{Z}[\sqrt{-3}][x]$ fails to be an HFD.

This example takes advantage of the fact that $\mathbb{Z}[\sqrt{-3}]$ is not integrally closed. This observation allows us to reduce the degree of the irreducible polynomial of ω after introducing appropriate factors (note the degree one polynomials on the left hand side are not monic). This argument has been extended to produce the following result ([20]).

Theorem 25. *[20, Theorem 2.2] Let $R[x]$ be an HFD, then the coefficient ring R must be integrally closed.*

This theorem has a corollary which serves to classify all Noetherian polynomial HFDs.

Corollary 26. *[20, Corollary 2.3] Let R be a Noetherian domain. The following conditions are equivalent:*

1) R is a Krull domain with $|Cl(R)| \leq 2$.

2) $R[x]$ is an HFD.

3) $R[x_1, ..., x_n]$ is an HFD for all $n \geq 1$.

Proof. 3) implies 2) is obvious. We will show that 1) implies 3) and 2) implies 1).

The first implication is due to Zaks [47]. Indeed if R is a Krull domain of class number not exceeding 2, $R[x_1, ..., x_n]$ is also a Krull domain of the same class number. In [47], Zaks showed that if R is a Krull domain then $R[x]$ is an HFD if and only if $|Cl(R)| \leq 2$. The implication follows inductively.

For the second implication, we assume that $R[x]$ is an HFD. Since R must be integrally closed (and Noetherian), R is a Krull domain. Hence $Cl(R[x]) = Cl(R) \leq 2$. □

The results above lead to a couple of interesting questions.

Question 27. If $R[x]$ is an HFD, is $R[x, y]$ an HFD?

The above result shows that the answer to this is positive in the Noetherian case. We conjecture an affirmative answer to this question in general.

Question 28. If $R[[x]]$ is an HFD, then is R integrally closed?

At first blush one would think that the answer to this is again positive. After all, polynomials tend to behave in a much nicer fashion than power series, so the restriction "integrally closed" (at least) should apply if $R[[x]]$ is an HFD. A closer look at out motivating example above:

$$(2x + (1 + \sqrt{-3}))(2x + (1 - \sqrt{-3})) = (2)(2)(x^2 + x + 1)$$

does not lead to an immediate contradiction as the element $x^2 + x + 1$ is a unit in $\mathbb{Z}[\sqrt{-3}][[x]]$. Admittedly, this by itself is not strong evidence that

$\mathbb{Z}[\sqrt{-3}][[x]]$ is an HFD, but some recent computations performed by the second author have shown that any irreducible in the UFD $\mathbb{Z}[\omega][[x]]$ can be thought of as an irreducible in $\mathbb{Z}[\sqrt{-3}][[x]]$, and perhaps this evidence is stronger. Indeed, if it is the case that $\mathbb{Z}[\sqrt{-3}][[x]]$ is an HFD (as it seems to be), then this would be quite surprising as the condition "integrally closed coefficient ring", though required for polynomial HFDs, would not be required for the characteristically ill-behaved power series extensions.

In closing, we would like to look at a generalization of polynomial extensions of HFDs that have a "$D + \mathfrak{M}$" flavor. As with the standard $D + \mathfrak{M}$ constructions, these prove to be a valuable source of examples.

Theorem 29 (Gonzalez). *[34, Proposition 1.8] Let $A \subseteq B$ be an extension which satisfies*

1) $U(B) \bigcap A = U(A)$ (where $U(R)$ is the unit group of R.)

2) Each irreducible element of A remains irreducible in B.

3) B is a UFD.

Then $A + xB[x]$ is an HFD.

The above result answers a question which was first posed in [12]. Two other papers have also offered answered to this question (D.F. Anderson and Nour El Abidine in [10] and Kim in [39]). We state the result as it appears in [34] because this form of it proves fruitful in generating examples of HFDs. In particular, we can glean the following.

Example 30. In the above theorem, let $A = \mathbb{Z}$ and $B = \mathbb{Z}[t]$. An easy verification of the hypotheses shows that $\mathbb{Z} + x\mathbb{Z}[t][x]$ is an HFD.

For a more exotic example, we consider the following also from [34].

Example 31. Consider the rings $A = \mathbb{Z}[\sqrt{85}] \subset \mathbb{Z}[\frac{1+\sqrt{85}}{2}] = B$. It can easily be shown that both A and B are HFDs that are not UFDs. It can be checked that the ring $A + xB[x]$ is an HFD. This is an interesting example, as it shows that HFDs can be constructed that are "polynomial-like" but not integrally closed. It also shows that the building blocks used (A and B) can have the minimal (HFD) condition.

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