# ON INTEGRAL DOMAINS WITH NO ATOMS

Jim Coykendall, Department of Mathematics, North Dakota State University, Fargo, North Dakota, 58105-5075

David E. Dobbs, Department of Mathematics, University of Tennessee, Knoxville, Tennessee, 37996-1300

Bernadette Mullins, Department of Mathematics and Statistics, Youngstown State University, Youngstown, Ohio, 44555-3302

**Abstract.** Antimatter domains are defined to be the integral domains which do not have any atoms. It is proved that each integral domain can be embedded as a subring of some antimatter domain which is not a field. Any fragmented domain is an antimatter domain, but the converse fails in each positive Krull dimension. A detailed study is made of the passage of the "antimatter" property between the partners within an overring extension. Special attention is given to characterizing antimatter domains in classes of valuation domains, pseudo-valuation domains, and various types of pullbacks.

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## 1. Introduction.

As evidenced by the book [1], much of the recent research on factorization in (commutative integral) domains has concerned atomic domains. Recall that if $r$ is a nonzero nonunit of a domain $R$, then $r$ is said to be an atom of (or to be irreducible in) $R$ if each factorization of $r$ in $R$ must exhibit a unit factor; and $R$ is said to be an atomic domain if each nonzero nonunit of $R$ is a product of finitely many atoms of $R$. Our concern in this paper is with domains whose factorization behavior differs greatly from that of atomic domains. To this end, we say that a domain $R$ is an *antimatter domain* if no element $r \in R$ is an atom of $R$. Of course, any field is an antimatter domain; more significantly, so is any fragmented domain. (Recall from [9] that a domain $R$ is called a fragmented domain if, for each nonzero nonunit $r \in R$, there exists a nonzero nonunit s of $R$ such that $r \in \cap_{n=0}^{\infty} Rs^n$.) The examples in Sections 2 and 3 show that the class of antimatter domains is more plentiful than that of the fragmented domains; in fact, Example 2.7 and Corollary 3.11 show that if $n$ is any positive integer or $\infty$, then there exists an antimatter domain of Krull dimension $n$ which is not fragmented. Nevertheless, it is convenient in this study to emphasize the same contexts which figured in [9], namely valuation domains, pseudo-valuation domains (in the sense of [14]), divided domains (in the sense of [7]), and various types of pullbacks.

Section 2 begins with the context of valuation domains. We note in Corollary 2.2 that each atomic valuation domain which is not a field must be Noetherian; that is, a DVR. (For arbitrary domains, we have the non-reversible implications Noetherian $\Rightarrow$ ACCP $\Rightarrow$ atomic; any non- Noetherian Krull domain satisfies ACCP, while Grams [13] has given an example of an atomic domain which does not satisify ACCP.) On the other hand, Proposition 2.3 (b) establishes that a valuation domain is antimatter if and only its value group (that is, the value group of any associated valuation), when written additively, has no least positive element. More generally, if $R$ is a domain with multiplicative group of units $U(R)$ and quotient field $K$, $R$ is an antimatter domain if and only if its group of divisibility $G(R) = (K \setminus \{0\})/U(R)$, when written additively, has no minimal positive elements. (Recall from [12] that $G(R)$ is partially ordered by decreeing that if $a, b \in K \setminus \{0\}$, then $aU(R) \leq bU(R)$ if and only if $b = ra$ for some $r \in R$. It is well known that if

$R$ is a valuation domain, then $G(R)$ is order-isomorphic to the value group of $R$. For additional motivation regarding a role for groups of divisibility in the present work, see the characterization of fragmented domains in [9, Remark 2.13 (a)].) The second half of Section 2 includes several examples of antimatter valuation domains showing, i.a., that an antimatter (resp., non-antimatter) domain may or may not have a proper antimatter overring other than its quotient field. (As usual, by an *overring* of a domain $R$, we mean any ring contained between $R$ and its quotient field.) Also noteworthy in Section 2 is the result (Proposition 2.13) that each domain can be embedded as a subring of an antimatter domain which need not be an overring.

Section 3 uses the examples of antimatter valuation domains from Section 2 to build examples of antimatter domains in more general contexts. The most important of these contexts concerns pseudo-valuation domains, or, in short, PVDs. (Recall from [14] that a domain $R$ is a PVD if $R$ has a – necessarily uniquely determined – valuation overring $T$ such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ as sets.) We show in Corollary 3.3(b) that a PVD is an antimatter domain if and only if its canonically associated valuation overring is an antimatter domain. (On the other hand, Corollary 3.3(a) establishes that a PVD is an atomic domain if and only if its canonically associated valuation overring is either a field or a DVR. Atomic PVDs have figured in factorization studies such as [3] and [5].) The above results on PVDs follow by reasoning based on the $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ condition. Now, the more general $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ condition can be characterized by using pullbacks [4, Theorem 3.25] and PVDs can also be canonically characterized by pullbacks [4, Proposition 2.6], and so we devote part of Section 3 to determining when pullbacks are antimattter domains. Perhaps our deepest result along these lines is Theorem 3.9, which characterizes a class of divided antimatter pullbacks. (Recall from [7] that a domain $R$ is a divided domain if each prime ideal $P$ of $R$ satisfies $P = PR_P$; that is, $P$ is comparable under inclusion with each ideal of $R$. Any PVD is a divided domain [8, p. 560].) In analyzing pullbacks in Section 3, we assume familiarity with associated gluing results on prime spectra and their order-theoretic consequences: cf. [11, Theorem 1.4 and Corollary 1.5].

For any domain $D$, it is convenient to let $D^*$ denote the set of nonzero elements of $D$, $U(D)$ the multiplicative group of units of $D$, $J(D)$ the Jacobson radical of $D$, and $\dim(D)$ the Krull dimension of $D$. Also, for any additively-written partially ordered abelian group $G$, we use $G^+$ to denote the set of positive elements of $G$. Any unexplained material is standard, as in [12].

## 2. Antimatter valuation domains and overring behavior.

For the sake of concreteness, we devote the first part of this section to antimatter valuation domains. To warm up, Corollary 2.2 characterizes the atomic valuation domains. For this, it is convenient to begin with the following useful dichotomy.

Proposition 2.1: Let $R$ be a valuation domain. Then either $R$ is an antimatter domain or $R$ contains (up to associates) exactly one atom. In the latter case, this atom is, in fact, a prime element of $R$ which generates the unique maximal ideal of $R$.

Proof: Suppose that $R$ is not an antimatter domain. Then $R$ contains at least one atom, say $r$. We claim that each nonunit $s$ of $R$ is divisible by $R$. Indeed, since $R$ is a valuation domain, either $r|s$ in $R$ or $s|r$ in $R$. If $r \not| \, s$, then $r = as$ for some $a \in R \setminus U(R)$, contradicting the irreducibility of $r$. This proves the above claim. Thus, if $r$ is an atom of $R$ and $M$ is the maximal ideal of $R$, then $M = Rr$. Then, since $r$ generates a nonzero prime ideal o f $R$, $r$ is necessarily a prime element of $R$. Finally, if $s$ is another atom of $R$, the above reasoning gives $M = Rs$, whence $Rs = Rr$, so that $s$ and $r$ are associated in $R$, to complete the proof. □

Corollary 2.2: For a valuation domain $R$, the following conditions are equivalent:

(1) $R$ is Noetherian (i.e., either a field or a DVR);
(2) $R$ satisfies ACCP;
(3) $R$ is an atomic domain.

Proof: As noted in the Introduction, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for any domain $R$. It remains only to prove that if $R$ is an atomic valuation domain, then $R$ is Noetherian. Without loss of generality, $R$ is not a field and so, by Proposi tion 2.1, $R$ has a unique atom up to associates, say $r$. Since $R$ is atomic, each element $s \in R^*$ can be written as $s = ur^n$ for some $u \in U(R)$ and some uniquely determined nonnegative integer $n = n(s)$. It follows easily that if $I$ is any non zero proper ideal of $R$, then $I = Rr^k$, where $k = \min\{n(s) : 0 \neq s \in I\}$. Thus, $R$ is a PID and, in particular, Noetherian. $\square$

We next characterize antimatter domains by using groups of divisibility.

Proposition 2.3: (a) Let $R$ be a domain with group of divisibility $G(R)$. Then $R$ is an antimatter domain if and only if $G(R)^+$ has no minimal elements.

(b) Let $R$ be a valuation domain with value group $G(R)$. Then $R$ is an antimatter domain if and only if $G(R)^+$ has no least element.

Proof: We recalled in the Introduction that the value group of a valuation domain is order-isomorphic to the group of divsibility of the domain. Accordingly, it suffices to prove (a). Let $K$ be the quotient field of $R$, and consider $a, b \in K^*$. Using the description of the group- and order-theoretic structures of $G(R)$ given in the Introduction, we may draw several conclusions in quick succession: $aU(R)$ is the identity element of $G(R) \Leftrightarrow a \in U(R)$; $aU(R)$ is nonnegative in $G(R) \Leftrightarrow a \in R^*$; $aU(R) \in G(R)^+ \Leftrightarrow a \in R^* \setminus U(R)$; and $aU(R) < bU(R)$ in $G(R)^+ \Leftrightarrow a, b \in R^* \setminus U(R), a|b$ in $R$, and $b \nmid a$ in $R$. It follows that $bU(R)$ is a minimal element of $G(R)^+$ if and only if $b \in R^* \setminus U(R)$ and $b$ is associated to each of its nonunit factors in $R$; that is, if and only if $b$ is an atom if $R$. Thus $G(R)^+$ has no minimal elements if and o nly if $R$ has no atoms; that is, if and only if $R$ is an antimatter domain. $\square$

Remark 2.4: (a) Recall from [9, Remark 2.13(b)] that a domain $R$ is fragmented if and only if its group of divisibility $G(R)$, when written additively, has the property that for each $g \in G(R)^+$, there exists $h \in G(R)^+$ such that $g \geq nh$ for each positive integer $n$. In the same spirit, we next note that Corollary 2.2 and Proposition 2.3(b) can be used to obtain the following classification result for valuation domains $R$:

(i) $R$ is a DVR (and atomic) $\Leftrightarrow$ $R$ is not a field but $R$ is Noetherian (and $G(R)^+$ has a least element);

(ii) $R$ is a nonatomic non-antimatter domain $\Leftrightarrow$ $R$ is not Noetherian and $G(R)^+$ has a least element;

(iii) $R$ is an antimatter domain $\Leftrightarrow$ $G(R)^+$ has no least element (and either $R$ is a field or $R$ is not Noetherian).

(b) The criteria in Proposition 2.3 are easy to apply. For instance, if a valuation domain $R$ has divisible value group, then $R$ is an antimatter domain. To see this, observe that a linearly ordered additively-written divisible group cannot have a least positive element and apply Proposition 2.3(b). Of course, valuation domains with divisible value group abound; for example, consider any valuation domain with value group $\mathbf{Q}$. (Cf. [12, Corollary 18.5].) However, the value group of an antimatter valuation domain need not be divisible, as in any example having value group $\mathbf{Z} \oplus \mathbf{Q}$ with the lexicographic order.

Recall from [9, Remark 2.7(b)] that an overring of a fragmented valuation domain need not be fragmented. With this result as motivation, we proceed to an example showing that "antimatter" exhibits similar instability behavior.

Example 2.5: There exists a fragmented (hence antimatter) valuation domain $R$ and a nonzero nonmaximal prime ideal $P$ of $R$ such that the overring $R_P$ is a DVR (hence atomic and not antimatter). For the construction, begin with a fragmented valuation domain $D$ which is not a field, for instance, the first valuation domain discussed in [9, Remark 2.7(a)]. Let $K$ be the quotient field of $D$ and put $T = K[[X]]$. Consider the pullback $R = T \times_K D = D + XT$. Since $T = K + XT$ is a valuation domain, we may apply the result in [9, Proposition 2.12(b)] on the classical $D + M$ construction to conclude that $R$ is fragmented (hence antimatter). Moreover, by [17, p. 35], the pullback $R$ is a valuation domain and $T = R_P$, where $P = XT$. As $T$ is a DVR, it suffices to note that $P$ is a nonmaximal prime ideal of $R$, that is, that $R \neq T$. This can be seen appealing to [9, Corollary 2.2(g)] or by noting that $R/P \cong D$ is not a field.

Next we pursue the theme introduced in Example 2.5 by developing a family of four valuation domain examples which exhibit the diversity of behavior of overring extensions in regard to the possible inheritance of the "antimatter" property.

Example 2.6: (a) There exists a two-dimensional antimatter valuation domain $R$ with quotient field $K$ such that there is no antimatter overring properly contained between $R$ and $K$.

(b) There exists a two-dimensional antimatter valuation domain $R$ whose one-dimensional overring is also an antimatter domain.

(c) There exists a two-dimensional valuation domain $R$ such that $R$ is not an antimatter domain but the one-dimensional overring of $R$ is an antimatter domain.

(d) There exists a two-dimensional valuation domain $R$ with quotient field $K$ such that $R$ is not an antimatter domain and there is no antimatter overring properly contained between $R$ and $K$.

Proof: Each of the domains $R$ constructed in (a)-(d) below will be shown to be a valuation domain whose value group is a lexicographic direct sum of two rank 1 abelian groups. Since the Krull dimension of a valuation domain coincides with the rank of its value group, it will follow that each of these domains $R$ is two-dimensional. Then since $R$ is a two-dimensional valuation domain, it will follow that the only proper overring of $R$ other than its quotient field is $R_P$, where $P$ is the height 1 prime ideal of $R$ (cf. [12, Theorem 17.6(a)]). Once the value group of $R_P$ has been identified, the later assertions regarding the antimatter or non-antimatter status of $R$ and $R_P$ will follow by applying the criteria in Proposition 2.3(b).

Notice that, apart from the requirement of two-dimensionality, the domain $R$ constructed via Nagata composition in Example 2.5 exhibits the other features asserted for (a). Rather than using Nagata composition throughout, we next develop (a)-(d) by constructing suitable localizations.

(a) Consider the domain $A = \mathbf{F}_2[\{X^\alpha, \frac{Y}{X^k} : \alpha \in \mathbf{Q}^+, k \geq 1\}]$ where $X$ and $Y$ are independent indeterminates over $\mathbf{F}_2$, the field with two elements. If $M$ is the ideal of $R$ generated by $\{X^\alpha, \frac{Y}{X^k} : \alpha \in \mathbf{Q}^+, k \geq 1\}$, then $M$ is a maximal ideal of $R$ since $R/M \cong \mathbf{F}_2$. Put $R = A_M$. The key point is that each nonzero nonunit of $R$ can be written either as a unit times $X^\alpha$ for some $\alpha \in \mathbf{Q}^+$ or as a unit times $\frac{Y^m}{X^\beta}$ for some $m \geq 1, \beta \in \mathbf{Q}$ (where $\beta$ may be positive, negative, or zero). It follows from routine calculations that $R$ is a valuation domain whose value group is $\mathbf{Z} \oplus \mathbf{Q}$ with the lexicographic order and that the prime spectrum of $R$ is $0 \subseteq P = (\{\frac{Y}{X^k} : k \geq 1\}) \subseteq M$. Evidently, $R_P$ has value group $\mathbf{Z}$ (and so $R_P$ is a DVR) and the assertions now follow by the reasoning indicated above.

(b) Now, consider the domain $A = \mathbf{F}_2[\{X^\alpha, \frac{Y^\beta}{X^k} : \alpha, \beta \in \mathbf{Q}^+, k \geq 1\}]$.

Again, let $R = A_M$, where $M$ is the maximal ideal $(\{X^\alpha, \frac{Y^\beta}{X^k} : \alpha, \beta \in \mathbf{Q}^+, k \geq 1\})$. In this case, every nonzero nonunit of $R$ can be written either as a unit times $X^\alpha$ for some $\alpha \in \mathbf{Q}^+$ or as a unit times $\frac{Y^\beta}{X^\gamma}$ for some $\beta \in \mathbf{Q}^+, \gamma \in \mathbf{Q}$. It follows that $R$ is a valuation domain whose value group is $\mathbf{Q} \oplus \mathbf{Q}$ with the lexicographic order and that the prime spectrum of $R$ is $0 \subseteq P = (\{\frac{Y^\beta}{X^k} : \beta \in \mathbf{Q}^+, k \geq 1\}) \subseteq M$. It is easily verified that $R_P$ has value group $\mathbf{Q}$.

(c) Next, consider $A = \mathbf{F}_2[\{X, \frac{Y^\beta}{X^k} : \beta \in \mathbf{Q}^+, k \geq 1\}]$ and let $R = A_M$, where $M = (\{X, \frac{Y^\beta}{X^k} : \beta \in \mathbf{Q}^+, k \geq 1\}) = (X)$. Then $R$ has value group $\mathbf{Q} \oplus \mathbf{Z}$ with the lexicographic order and $R_P$ has value group $\mathbf{Q}$, where $P = (\{\frac{Y^\beta}{X^k} : \beta \in \mathbf{Q}^+, k \geq 1\})$ is the height 1 prime ideal of $R$.

(d) Finally, consider $A = \mathbf{F}_2[\{X, \frac{Y}{X^k} : k \geq 1\}]$ and let $R = A_M$, where $M = (\{X, \frac{Y}{X^k} : k \geq 1\}) = (X)$. Then $R$ has value group $\mathbf{Z} \oplus \mathbf{Z}$ with the lexicographic order and $R_P$ has value group $\mathbf{Z}$, where $P = (\{\frac{Y}{X^k} : k \geq 1\})$ is the height 1 prime ideal of $R$. $\square$

Next, we continue building examples of antimatter valuation domains by showing that nonfragmented examples exist in all positive dimensions.

Example 2.7: If $n$ is any positive integer or $\infty$, then there exists an $n$-dimensional antimatter valuation domain which is not fragmented.

Proof: (a) Consider the domain $A = \mathbf{F}_2[\{X^\alpha, \frac{Y_1}{X^k}, \frac{Y_2}{Y_1^k}, \ldots, \frac{Y_{n-1}}{Y_{n-2}^k} : \alpha \in \mathbf{Q}^+, k \geq 1\}]$ or, written more succinctly, $A = \mathbf{F}_2[\{X^\alpha, \frac{Y_m}{Y_{m-1}^k} : \alpha \in \mathbf{Q}^+, k \geq 1, 1 \leq m \leq n-1\}]$. Let $R = A_M$, where $M$ is the maximal ideal generated by $\{X^\alpha, \frac{Y_m}{Y_{m-1}^k} : \alpha \in \mathbf{Q}^+, k \geq 1, 1 \leq m \leq n-1\}$. By determining the possible forms for the nonzero nonunits of $R$ as in the proof of Example 2.6, we see that $R$ is an $n$-dimensional antimatter valuation domain whose value group is $\mathbf{Z} \oplus \mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$ with the lexicographic order and the prime spectrum of $R$ is $0 \subseteq (\{\frac{Y_{n-1}}{Y_{n-2}^k} : k \geq 1\}) \subseteq (\{\frac{Y_{n-2}}{Y_{n-3}^k}, \frac{Y_{n-1}}{Y_{n-2}^k} : k \geq 1\}) \subseteq \cdots \subseteq M$. However, $R$ is not a fragmented domain since $X$ is a nonzero nonunit of $R$ which does not lie in $\cap_{n=0}^{\infty} Rs^n$ for any nonunit $s \in ! R$.

(b) The above construction may be extended to an infinite-dimensional example by considering the domain $A = \mathbf{F}_2\ [\{X^\alpha, \frac{Y_1}{X^k}, \frac{Y_2}{Y_1^k}, \frac{Y_3}{Y_2^k}, \ \ldots\ : \alpha \in \mathbf{Q}^+, k \geq 1\}] = \mathbf{F}_2\ [\ X^\alpha, \frac{Y_m}{Y_{m-1}^k} : \alpha \in \mathbf{Q}^+, k \geq 1, m \geq 1\}]$. Let $R = A_M$, where

$M$ is the maximal ideal generated by $\{X^\alpha, \frac{Y_m}{Y_{m-1}^k} : \alpha \in \mathbf{Q}^+, k \geq 1, m \geq 1\}$. By reasoning analogous to the a bove, $R$ is an infinite-dimensional antimatter valuation domain which is not a fragmented domain. $\square$

In view of Example 2.6, it seems natural to ask when an antimatter valuation domain has a proper antimatter overring other than its quotient field. An answer is given in Proposition 2.9, with "valuation" replaced by the more general "divided" concept. For this, we need a lemma which shows that, despite Example 2.7, *any* antimatter valuation domain exhibits *some* "fragmented" behavior. It is convenient to say that if $R$ is a domain and $r$ is a nonzero nonunit of $R$, then $r$ *fragments in* $R$ if there exists $s \in R \setminus U(R)$ such that $r \in \bigcap_{n=0}^{\infty} Rs^n$. Clearly, a domain $R$ is a fragmented domain if and only if each nonzero nonunit of $R$ fragments in $R$.

Lemma 2.8: Let $R$ be a divided antimatter domain. Let $r$ be a nonzero element of a prime ideal $P$ of $R$. Then either $r$ fragments in $R$ or $r$ is not an atom of $R_P$.

Proof: It suffices to show that if $r$ is an atom of $R_P$, then $r$ fragments in $R$. Since $R$ is antimatter, $r$ is not an atom of $R$, and so $r = ab$ for some $a, b \in R \setminus U(R)$. However, $r$ is an atom of $R_P$, and so we may assume, without l oss of generality, that $b \in U(R_P)$. Then $b \in U(R_P) \cap R = R \setminus P$ and, since $P$ is a prime ideal of $R$, it follows from $ab = r \in P$ that $a \in P$. Since $b^n \in R \setminus P$ for each positive integer $n$, the "divided" property o f $R$ yields that $a(b^n)^{-1} \in PR_P = P \subseteq R$, whence $a \in Rb^n$ and $r = ab \in Rb^{n+1}$. In particular, $r$ fragments in $R$. $\square$

Proposition 2.9: Let $R$ be a divided antimatter domain properly contained in its quotient field $K$. If $R$ has no antimatter overrings properly contained between $R$ and $K$, then for every nonzero nonmaximal prime ideal $P$ of $R$, there exists a nonzero element $r \in P$ which fragments in $R$.

Proof: Assume that $R$ has no antimatter overrings strictly between $R$ and $K$. Then for any nonzero nonmaximal prime $P$, there exists an atom $r$ in $R_P$, and we may assume that $r \in P$. By applying Lemma 2.8, we see that $r$ fragments in $R$. $\square$

Remark 2.10: (a) A two-dimensional example of a domain $R$ satisfying the hypothesis in Proposition 2.9 is given by the ring $R$ constructed in Example 2.6(a). In this example, $Y$ can play the role of $r$, since $Y \in \cap_{n=0}^{\infty} RX^n$.

(b) The converse of Proposition 2.9 is false, even for fragmented valuation domains. To see this, consider the linearly ordered set X consisting of $a_0 < a_1 < a_2 < \cdots < a_n < \cdots < a_\infty = b_0 < b_1 < b_2 < \cdots < b_n < \cdots < b_\infty$. It is easy to verify that every chain in X has a supremum and an infimum; and that X satisfies the "im mediate neighbors" property. In other words, X satisfies Kaplansky's properties (K1) and (K2), as described in [15, p. 429]. Therefore, by [15, Corollary 3.6], there exists a valuation domain $R$ whose prime spectrum is order-isomorphic to X. Since the maximal ideal $M$ of $R$ corresponds to $b_\infty \in X$, $M$ is the union of the nonmaximal prime ideals of $R$. As $R$ is a divided domain, it now follows from [9, Theorem 2.5] that $R$ is fragmented. Thus, by the above remarks, each nonzero nonunit of $R$ fragments in $R$. On the other hand, if $P$ is the prime ideal of $R$ which corresponds to $a_\infty = b_0 \in X$, then $R_P$ is a proper overring of $R$ which is an antimatter domain. Indeed, since $PR_P = P$ is the union of the nonmaximal prime ideals of the valuation domain $R_P$, it follows by the above reasoning that $R_P$ is actually a fragmented domain.

Example 2.6(a) exhibited an antimatter valuation domain with a proper overring which is not an antimatter domain. This raises the question of where atoms can be found within an overring extension of a given antimatter domain. The next result answers this question in the case of an integral overring. More generally, observe that the lying-over property of integral extensions (cf. [12, Theorem 11.5]) ensures that if $R \subseteq T$ is an integral extension of domains, then $U(T) \cap R = U(R)$.

Proposition 2.11: Let $R \subseteq T$ be domains such that $R$ is an antimatter domain. Suppose also that $U(T) \cap R = U(R)$ (for instance, $T$ integral over $R$). Then no element of $R$ is an atom of $T$.

Proof: Deny. Choose an element $r \in R \setminus U(R)$ such that $r$ is an atom of $T$. Since $R$ is antimatter, $r$ is not an atom of $R$, and so $r = ab$ for some $a, b \in R \setminus U(R)$. However, the hypothesis $U(T) \cap R = U(R)$ ensures that $RinusU(R) \subseteq T \setminus U(T)$, whence the factorization $r = ab$ reveals that $r$ is not an atom of $T$, the desired contradiction. $\square$

Remark 2.12: One cannot delete the integrality-like hypothesis regarding units in Proposition 2.11. To see this, consider the antimatter domain $R$ constructed in Example 2.6(a). Recall that $T = R_P$ is a DVR with local uniformizing paramater $Y$. Thus, $Y$ is an atom of $T$ and, of course, $Y = X \cdot \frac{Y}{X} \in R$.

Despite parts (b) and (d) of Example 2.6, some domains do not have antimatter overrings other than their quotient field. For instance, if $R$ is a one-dimensional Noetherian domain and if $T$ is an overring of $R$ distinct from the quotient field of $R$, then $T$ is Noetherian by the Krull-Akizuki Theorem (cf. [6, Proposition 5, p.500]); it follows that $T$ is atomic and hence not an antimatter domain. Accordingly, we close the section by considering passage of the "antimatter" property *beyond* the quotient field. The situation here is remarkably simpler than in the overring context studied in Example 2.6.

Theorem 2.13: Every domain can be embedded as a subring of some antimatter domain which is not a field.

Proof: Let $R$ be a domain with quotient field $K$. There are two cases: either $R = K$ is a field or $R$ is not a field. Suppose $R = K$. Consider the domain $A = K[\{X^\alpha : \alpha \in \mathbf{Q}^+\}]$ and its maximal ideal $M$ generated by $\{X^\alpha : \alpha \in \mathbf{Q}^+\}$. It is easy to see that each nonzero nonunit of $T = A_M$ can be written as $uX^\alpha$, for some $u \in U(T)$ and $\alpha \in \mathbf{Q}^+$. It follows that $T$ is a valuation domain with value group $\mathbf{Q}$. Hence, by Corollary 2.3(b), $T$ is an antimatter domain.

It remains to consider the case in which $R$ is not a field. Let $\bar{K}$ be an algebraic closure of $K$ and let $R^a$ be the integral closure of $R$ in $\bar{K}$. Of course, $R \subseteq R^a$ and by integrality (cf. [12, Lemma 11.3]), $R^a$ is not a f ield. It suffices to show that $R^a$ is an antimatter domain. For this, consider any nonzero nonunit $\alpha \in R^a$, let $n \geq 2$ be a positive integer, and take $\beta \in \bar{K}$ to be any root of the polynomial $X^n - \alpha$. As $\beta \in \bar{K}$, the quotient field of $R^a$, and $\beta$ is integral over the integrally closed domain $R^a$, we have $\beta \in R^a$. Since $\alpha = \beta^n$, it follows that $\alpha$ is not an atom of $R^a$. As $\alpha$ is arbitrary, $R^a$ is antimatter. $\square$

## 3. Antimatter domains in the context of pseudo-valuation domains and other pullbacks.

To broaden the applications of antimatter domains, we identify the antimatter PVDs in Corollary 3.3(b). In studying PVDs, it is convenient to work in the more general context of distinct domains $R \subset T$, not fields, such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ as sets. Recall from [4, Proposition 3.3] that under these conditions, $R$ is quasilocal, say with maximal ideal $M$; an oft-used consequence is that $R \setminus U(R) = M = T \setminus U(T)$. For this "equal spectra" context, the atomic domains are as easy to characterize as the antimatter domains: see Proposition 3.2. We begin with a useful result on atoms.

Lemma 3.1: Let $R \subseteq T$ be domains such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. Let $r \in R$. Then $r$ is an atom of $R$ if and only if $r$ is an atom of $T$.

Proof: Without loss of generality, $r \neq 0$ and $R \neq T$. By the above remarks, $R$ is quasilocal with maximal ideal $M = R \setminus U(R) = T \setminus U(T)$. Without loss of generality, $r \in M$. Now, $r$ is not an atom of $R \Leftrightarrow r = ab$ f or some $a, b \in M \Leftrightarrow r = ab$ for some $a, b \in T \setminus U(T) \Leftrightarrow r$ is not an atom of $T$. The assertion is now immediate. □

It was shown recently by Badawi [5, Theorem 2] that any atomic PVD is a half-factorial domain. (The same conclusion may be reached by combining [3, Theorem 6.2 and Corollary 5.2] with [4, Proposition 2.5] and Corollary 3.3(a).) Corollary 3.3(a) characte rizes the atomic PVDs. First, in Proposition 3.2, we work in the "equal spectra" context. Notice that Proposition 3.2(b) is the "antimatter" analogue of a "fragemented" result [9, Proposition 2.12(a)].

Proposition 3.2: Let $R \subseteq T$ be domains such that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$. Then:
(a) $R$ is an atomic domain if and only if $T$ is an atomic domain;
(b) $R$ is an antimatter domain if and only if $T$ is an antimatter domain.

Proof: Any atom of $T$ must be a nonunit of $T$ and hence, since $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, must be an element of $R$. Since a domain is antimatter if and only if it has no atoms, (b) now follows directly form Lemma 3.1. As for (a), $R \neq T$, without loss of generality. Then $R$ is quasilocal, say with maximal ideal $M = R \setminus U(R) = T \setminus U(T)$. By Lemma 3.1 and the first sentence of this proof, "atom of $R$" is equivalent to "atom of $T$". Hence, $R$ is an atomic domain $\Leftrightarrow$ fo r each nonzero $r \in M$, $r$ is a product of finitely many atoms of $R \Leftrightarrow$ for each nonzero $r \in M$, $r$ is a product of finitely many atoms of $T \Leftrightarrow$ $T$ is an atomic domain. $\square$

Corollary 3.3: Let $R$ be a pseudo-valuation domain, with canonically associated valuation overring $V$. Then:

(a) $R$ is an atomic domain if and only if $V$ is Noetherian (that is, if and only if $V$ is either a field or a DVR).

(b) $R$ is an antimatter domain if and only of $V$ is an antimatter domain (that is, if and only if $G(V)^+$ has no least element).

Proof: $V$ is the valuation overring of $R$ satisfying $\text{Spec}(R) = \text{Spec}(V)$. Accordingly, (a) follows from Proposition 3.2(a) and Corollary 2.2; and (b) follows from Proposition 3.2(b) and Proposition 2.3(b). $\square$

Corollary 3.4: (Bawadi [5, Theorem 7]) If $R$ is an atomic pseudo-valuation domain, then $\dim(R) \leq 1$.

Proof: Let $V$ be the canonically associated valuation overring of $R$. By Corollary 3.3(a), either $V$ is a field or $V$ is a DVR. In the former case, $\dim(V) = 0$; and in the latter case, $\dim(V) \leq 1$. Hence $\dim(V) \leq 1$. Howeve r, since $\text{Spec}(R) = \text{Spec}(V)$, we have $\dim(R) = \dim(V)$, to complete the proof. $\square$

The reader may find it interesting to compare the above "equal-spectra" approach to atomic PVDs with the factorization-theoretic approaches using length functions in [3] and [5].

Remark 3.5: We next sketch a proof of Corollary 3.3(b) which seems rather different from the proof given above. Let $V$ be the canonically associated valuation overring of a PVD, $R$. Let $M$ be the maximal ideal of $R$ (and $V$), and consider the fields $k = R/M$ and $F = V/M$. By [16, (5), page 397], there is a lexicographically exact sequence of multiplicative groups

$$1 \longrightarrow F^*/k^* \longrightarrow G(R) \longrightarrow G(V) \longrightarrow 1.$$

Since the partial order on $F^*/k^*$ is the trivial (identity) partial order (cf. [16, page 389]), it follows from the definition of a lexicographically exact sequence [19, page 577] that $G(R)^+ = \{bU(R) : b$ is in the quotient field of $R$ and $bU(V) \in G(V)^+\}$. Thus, $G(R)^+ = \{bU(R) : b \in R^* \setminus U(R)\} = \{bU(R) : 0 \neq b \in M\}$. (Of course, the same conclusion can be found directly, by reasoning as in the proof of Proposition 2.3.) Similarly, $G(V)^+ = \{bU(R) : 0 \neq b\imath n M\}$. Let $\beta = bU(R)$ (resp., $bU(V)$) be a member of $G(R)^+$ (resp., $G(V)^+$). By reasoning as in the proof of Proposition 2.3, we see that $\beta$ is a minimal positive element in $G(R)$ (resp., $G(V)$) if and only if there does not exist a nonzero element $a \in M$ such that $b = am$ for some element $m \in M$. In particular, $G(R)^+$ has a minimal element if and only if $G(V)^+$ has a minimal element. Thus, by the criterion in Proposition 2.3(a), $R$ is an antimatter domain if and only if $V$ is an antimatter domain, to complete the proof.

It is convenient next to record the following useful result, an "antimatter" analogue of a "fragmented" result [9, Lemma 2.3].

Proposition 3.6: Let $R$ be an antimatter domain and let $P$ be a prime ideal of $R$ such that $P \subseteq J(R)$. Then $R/P$ is an antimatter domain.

Proof: Set $A = R/P$. If the assertion fails, choose an atom $\alpha$ of $A$. Then $\alpha = r + P$ for some $r \in R$. As $\alpha$ is a nonzero nonunit of $A$, we see that $r \in R \setminus (P \cup U(R))$. Since $R$ is an antimatter domain, $r$ is not an atom of $R$, and so $r = bc$ for some $b, c \in R \setminus U(R)$. Put $\beta = b + P$ and $\gamma = c + P$. Then $\alpha = \beta\gamma$ in $A$. As $\alpha$ is an atom of $A$, we may assume, without loss of generality, that $\beta \in U(A)$. Hence, there exists a coset representative $s \in R$ of $\beta^{-1}$; that is, $\beta(s + P) = 1 + R \in A$. It follows that $bs - 1 \in P \subseteq J(R)$, whence $\beta s \in 1 + J(R) \subseteq U(R)$ and $b \in U(R)$, the desired contradiction. $\square$

It was shown in [4, Proposition 2.6] that PVDs may be characterized as pullbacks of the form $R = V \times_F k$, where $(V, M)$ is a valuation domain with residue field $F = V/M$ and $k$ is a subfield of $F$. (Then $V$ is the canonically associated valuation overring of $R$ and $k \cong R/M$.) As the "equal spectra" context may also be characterized using pullbacks [4, Theorem 3.25], we expand upon our above work by devoting the rest of this paper to studying pullbacks which are antimatter (or, occasionall y, atomic) domains. We begin this direction with what amounts to a pullback-theoretic reformulation of Proposition 3.2.

Proposition 3.7: Let $(T, M)$ be a quasilocal domain, let $F = T/M$, and let $D$ be a subfield of $F$. Consider the pullback $R = T \times_F D$. Then:
(a) $R$ is an atomic domain if and only if $T$ is an atomic domain.
(b) $R$ is an antimatter domain if and only if $T$ is an antimatter domain.

Proof: Applying the gluing result [11, Theorem 1.4] to the pullback defining $R$, we see that $\mathrm{Spec}(R)$, with the Zariski topology, is homeomorphic to the quotient space of the disjoint union of $\mathrm{Spec}(T)$ and $\mathrm{Spec}(D)$ in which $M \in \mathrm{Spec}(T)$ is identif ied with $0 \in \mathrm{Spec}(D)$. It follows that the canonical map $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$ is a homeomorphism, whence $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ as sets. (We have given the above argument as a gentle reminder of the gluing techniques from [11]. These will be needed in our later results whenever $D$ is not assumed to be a field, for then $\mathrm{Spec}(R) \neq \mathrm{Spec}(T)$. For the *present* result, since $D$ *is* a field, we may obtain the conclusion that $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ more simply than quoting [11], as follows. Observe that $M$ is a maximal ideal of $R$, since $R/M \cong D$ is a field. Thus, each maximal ide al of $T$ (namely, $M$!) is a maximal ideal of $R$ and so, by [4, Theorem 3.10], $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ as sets.) As $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, an application of Proposition 3.2 completes the proof. □

We show next that the condition that $D$ is a field (which was a hypothesis in Proposition 3.7) is actually a consequence of assuming that the pullback $R$ is an atomic domain. Theorem 3.8 was obtained in [2, Proposition 1.2] for the case $T = F + M$ (but for the result in [2], $T$ was not assumed quasilocal).

Theorem 3.8: Let $(T, M)$ be a quasilocal domain which is not a field, let $F = T/M$, and let $D$ be a subring of $F$. Consider the pullback $R = T \times_F D$. Then $R$ is an atomic domain if and only if $T$ is an atomic domain and $D$ is a field.

Proof: By Proposition 3.7(a), it suffices to show that if $R$ is atomic, then $D$ is a field. Deny. Using the gluing results from [11], observe that as a partially ordered set, $\mathrm{Spec}(R)$ can be obtained (up to order-isomorphism) by gluing $\mathrm{Spec}(D)$ "on t op of" $\mathrm{Spec}(T)$ with $0 \in \mathrm{Spec}(D)$ identified with $M \in$

Spec($T$). Now, since $D$ is not a field, we can choose a nonzero maximal ideal $P$ of $D$. Under the above gluing, $P$ corresponds to a maximal ideal $Q$ of $R$ such that $M$ is properly contained in $Q$. (In detail, if $\pi : T \longrightarrow F$ is the canonica l surjection, then $Q = \pi^{-1}(P)$.) Thus, by picking $r \in Q \setminus M$, we produce an element $r \in R \setminus U(R)$ such that $r \notin M$. Hence, $r \in R \setminus M \subseteq T \setminus M = U(T)$, so that $r^{-1} \in T$. Thus, if $m \in M$, we hav e $m = r(r^{-1}m)$, with $r^{-1}m \in TM = M \subseteq R \setminus U(R)$. In particular, $M$ contains no atoms of $R$. However, since $M \neq 0$, we may choose a nonzero element $s \in M$. Then, since $R$ is atomic, factor $s$ as a product of atoms $\alpha_i$ o f $R$. Now, since $R/M \cong D$ is a domain, $M$ is a prime ideal of $R$, and so some $\alpha_i \in M$, contradicting the fact that $M$ contains no atoms of $R$. The proof is complete. □

One cannot delete the hypothesis in Theorem 3.8 that $T$ is not a field. Indeed, if $M = 0$, then $F = T$ and $R = D$, so that a counterexample to the "only if" assertion would result by taking $T$ to be a field and $D$ be an atomic domain such that $DeteqT$ and $D$ is not a field.

Theorem 3.9 is an "antimatter" counterpart of Theorem 3.8 and can be viewed as a vast generalization of the "if" assertion in Corollary 3.3(b). (More specifically, if $R$ is a PVD, then $D$ is a field possibly distinct from $F$, but the proof of Theo rem 3.9 adapts to this case since $MR_M = M$.)

Theorem 3.9: Let $(T, M)$ be a divided antimatter domain, let $F = T/M$, and let $D$ be a domain with quotient field $F$. Consider the pullback $R = T \times_F D$. Then $R$ is a divided antimatter domain if and only if $D$ is a divided antimatter domain.

Proof: Any divided domain is quasilocal [7, Proposition 2.1], and so $T$ does indeed have a unique maximal ideal, say $M$. By using gluing techniques from [11] as in the proof of Theorem 3.8, we see that $M \subseteq J(R)$. Thus, by Proposition 3.6, if $R$ is an antimatter domain, then so is $R/M \cong D$. Moreover, if $R$ is a divided domain, then [7, Lemma 2.2(c)] assures that $R/M \cong D$ is also a divided domain, thus completing the proof of the "only if" assertion. (Notice that the "only if " pro of did not require that $D$ have quotient field $F$ or that $T$ be divided and antimatter, but only that $(T, M)$ be quasilocal. The full hypotheses on $D$ and $T$ will be used in the "if" proof below.)

For the "if" assertion, suppose that $D$ is a divided antimatter domain.

A well known calculation in the folklore of pullbacks shows that $R_M = T$ follows from the condition that $F$ is the quotient field of $D$. Hence, $MR_M = MT = M$. As $R/M \cong D$ and $R_M = T$ are each divided domains, [10, Proposition 2.12] yields that $R + MR_M = R + M = R$ is also a divided domain. It remains only to show that $R$ is an antimatter domain.

If the assertion fails, choose an atom $r$ of $R$. Since $T$ is an antimatter domain, $T$ has no atoms and, in particular, $r$ is not an atom of $T$. If $r \in M$, then $r$ is a nonunit of $T$, and so $r = t_1t_2$ for some elements $t_1, t_2 \in T \setminus U(T) = M \subseteq R \setminus U(R)$, contradicting the irreducibility of $r$ in $R$. Hence $r \notin M$.

Let $\pi : T \longrightarrow T/M = F$ denote the canonical surjection, and put $d = \pi(r)$. As $r \in R \setminus M$, we have $d \neq 0$. In fact, $d \notin U(D)$. To see this, pick a maximal ideal $N$ of $R$ such that $r \in N$, observe that $M \subseteq J(R) \subseteq N$, and (identifying $D$ with $R/M$ via $\pi$) conclude that $d = r + M$ is an element of the maximal ideal $N/M$ of $D$. (We pause to give an alternate proof that $d$ is a nonunit of $D$. Deny. Then there exists a coset representative $s \in R$ such that $rs - 1 \in M$, whence $rs \in 1 + M \subseteq 1 + J(R) \subseteq U(R)$ and $r \in U(R)$, contradicting that $r$ is an atom of $R$.)

Since $D$ is an antimatter domain, $d$ is not an atom of $D$. As $d \in D^* \setminus U(D)$, there exist $r_1, r_2 \in R$ such that $d = \pi(r_1)\pi(r_2)$ and $\pi(r_1), \pi(r_2) \in D^* \setminus U(D)$. It follows that $r_1, r_2 \in R \setminus (MpU(R))$. However, since $d = \pi(r_1)\pi(r_2)$, we have $r - r_1r_2 \in M$, so that

$$r(r_1r_2)^{-1} - 1 = (r - r_1r_2)(r_1r_2)^{-1} \in MR_M = M.$$

Then $u = r(r_1r_2)^{-1}$ satisfies $u \in 1+M \subseteq 1+J(R) \subseteq U(R)$, so that $r = ur_1r_2$, contradicting the irreducibility of $r$ in $R$, to complete the proof. □

We close with some applications of Theorem 3.9. The first of these concerns the classical $D + M$ construction (in the sense of [12]).

Corollary 3.10: Let $V = F + M$ be a nontrivial valuation domain, where $F$ is a field and $M$ is the maximal ideal of $V$. Let $D$ be a domain with quotient field $F$, and put $R = D + M$. Then $R$ is a divided antimatter domain if and only if $D$ is a divided a ntimatter domain.

Proof: As recalled in the Introduction, any (pesudo-)valuation domain is a divided domain. Since $V \times_F D = D + M = R$, an application of Theorem

3.9 completes the proof. $\square$

Finally, we show how to use Theorem 2.9 to construct non-fragmented antimatter valuation domains of arbitrary finite positive Krull dimension. In contrast to the constructions in Example 2.7(a), whose properties involved extensive calculations which were left to the reader, the pullback methods in Corollary 3.11 lead to simple proofs. However, we have not found an infinite-dimensional pullback-theoretic construction with the properties in Example 2.7(b).

Corollary 3.11: (a) Let $K$ be a field, let $A = K[\{X^\alpha : \alpha \in \mathbf{Q}^+\}]$, let $M$ be the ideal of $A$ generated by $\{X^\alpha : \alpha \in \mathbf{Q}^+\}$, and put $R = A_M$. Then $R$ is a one-dimensional antimatter valuation domain with residue field $K$.

(b) For each positive integer $n$, an $n$-dimensional non-fragmented antimatter valuation domain $V_n$ may be inductively constructed as follows. For $n = 1$, take $V_1$ to be the ring $R$ constructed in (a), using $K = \mathbf{F}_2$. For $n \geq 1$, given $V_n$ as asserted with quotient field $K_n$, define $V_{n+1}$ as the pullback $R \times_{K_n} V_n$, where $R$ is the ring constructed in (a), using $K = K_n$.

Proof: (a) If $p, q \in \mathbf{Z}^+$, then $(X^{\frac{p}{q}})^q = X^p \in K[X]$. It follows easily that $A$ is integral over $K[X]$, whence $\dim(A) = \dim(K[X]) = 1$ (cf. [12, 11.8]). As $1 \notin M$, $M$ has height 1 in $A$, and so $\dim(R) = 1$. Of course, the residue field of $R$ is canonically $A_M/MA_M \cong A/M \cong K$. The remaining details are like those left to the reader for the case $n = 1$ in Example 2.7(a). For the sake of completeness (and because $K$ need not be $\mathbf{F}_2$ here), we sketch the details next. Reasoning as in Example 2.6(a), we see that each nonzero nonunit of $R$ can be written as $uX^\alpha$, for some $u \in U(R)$ and $\alpha \in \mathbf{Q}^+$. It follows readily that $R$ is a valuation domain. Moreover, as $X^{ha} = (X^{\frac{\alpha}{2}})^2$ for each $\alpha \in \mathbf{Q}^+$, $R$ has no atoms; that is, $R$ is an antimatter domain.

(b) The "non-fragmented" conclusions follow from the result (cf. [9, Corollary 2.6 or Corollary 2.8]) that a fragmented valuation domain which is not a field must have infinite Krull dimension. Thus, by (a), $V_1$ has the asserted properties. (Another w ay to see that $V_1$ is not fragmented is to recall that any one-dimensional domain is Archimedean [18, Corollary 1.4] and, hence, not fragmented [9, page 464].)

Now, suppose that $n \geq 1$ and that $V_n$, with quotient field $K_n$, has the asserted properties. As $V_{n+1} = R \times_{K_n} V_n$ is obtained by Nagata composition of valuation domains, it follows from [17, page 35] that $V_{n+1}$ is a valuation doma in. Moreover, since $\dim(R) = 1$ and $\dim(V_n) = n$, it follows from standard gluing techniques [11, Proposition 2.1(5)] that $\dim(V_{n+1}) = \dim(V_n) + \dim(R) = n + 1$. Finally, since (pseudo-)valuation domains are divided, Theo rem 2.9 may be applied to show that $V_{n+1}$ is an antimatter domain. The proof is complete. □

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Jim Coykendall, Department of Mathematics, North Dakota State University, Fargo, North Dakota, 58105-5075

David E. Dobbs, Department of Mathematics, University of Tennessee, Knoxville, Tennessee, 37996-300

Bernadette Mullins, Department of Mathematics and Statistics, Youngstown State University, Youngstown, Ohio, 44555-3302