SURVIVAL-PAIRS OF COMMUTATIVE RINGS HAVE THE LYING-OVER PROPERTY

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Abstract

Let $R \subseteq T$ be a unital extension of commutative rings. It is proved that $(R, T)$ is a lying-over pair (in the sense that $A \subseteq B$ satisfies LO for all rings $R \subseteq A \subseteq B \subseteq T$) if (and only if) $(R, T)$ is a survival-pair (in the sense that $P \cap T \neq T$ for all rings $R \subseteq A \subseteq T$ and all prime ideals $P$ of $A$).

As a consequence, $T$ is integral over $R$ if (and only if) $(R[X], T[X])$ is a survival-pair, where $X$ is an indeterminate over $T$. It is also proved that a unital homomorphism of commutative rings satisfies LO if and only if it is universally a survival homomorphism.

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1 INTRODUCTION

All rings considered are commutative with identity; all inclusions of rings and all ring homomorphisms are unital. As in [6], if $\varphi$ is a property of some ring extensions, we say that $(R, T)$ is a $\varphi$-pair in case $R \subseteq T$ are rings such that $A \subseteq B$ satisfies $\varphi$ for all rings $R \subseteq A \subseteq B \subseteq T$. The “pair” approach figures in the result that helped to motivate both [5] and [6], namely, [6, Folklore Theorem, p. 454]: a ring extension $R \subseteq T$ is integral if and only if $(R, T)$ is...
both an INC-pair and an LO-pair. (Following [12, p. 28], we let INC, LO, and GU denote the incomparable, lying-over, and going-up properties, respectively. In particular, a ring extension \( A \subseteq B \) satisfies LO if and only if each prime ideal \( P \) of \( A \) is of the form \( Q \cap A \) for at least one prime ideal \( Q \) of \( B \).) Over the years, INC-pairs have become well-understood. They were introduced, without the terminology, in [5, Theorem, p. 38], which implies that \((R, T)\) is an INC-pair if and only if \( R \subseteq T \) are rings such that each element of \( T \) is a root of a polynomial over \( R \) having a unit coefficient. In other words, we have [5, Corollary 4]: \((R, T)\) is an INC-pair if and only if \( R \subseteq T \) is a P-extension (in the sense of Gilmer-Hoffmann [10]). In addition, although the implication \( \text{INC} \Rightarrow \text{MINC} \) is non-reversible, the concepts of INC-pair and MINC-pair are equivalent [6, Example 2.2, Corollary 2.4(bis)]. More recently, Ayache-Jaballah [1] showed that INC-pairs are the same as the residually algebraic pairs. For a survey of current knowledge about INC-pairs, see [9, Section 6.5]. The present paper contributes to deepening our understanding of LO-pairs.

LO-pairs were introduced in [6]. Subsequent work on this topic has included a study of LO-pairs of affine algebras [13] and a recent non-integral example [2] that answered a question raised in [6, Remark 3.12(b)]. Much of [6] was motivated by the following non-reversible implications connecting properties of ring extensions: \( \text{GU} \Rightarrow \text{LO} \Rightarrow \text{survival} \). (Following [12, p. 35], we say that if \( R \subseteq T \) are rings and \( I \) is a proper ideal of \( R \), then \( I \) survives in \( T \) if \( IT \neq T \), that is, if \( 1 \notin IT \); and that \( R \subseteq T \) is a survival extension if each proper (resp., prime) ideal of \( R \) survives in \( T \).) In the spirit of [6, Corollary 2.4(bis)], it was shown in [6, Corollary 3.2] that LO-pairs are the same as GU-pairs. It seems irresistible to ask if they are also the same as the survival-pairs. Our main result, Theorem 2.2, settles the matter.

Although each LO-pair must be a survival-pair, the converse does not seem obvious to us. In fact, although the property of being an LO-pair is easily seen to be a local property [6, Lemma 2.11(c)], we do not consider it to be obvious whether “survival-pair” is a local property. Nevertheless, there has long been compelling evidence suggesting that survival-pairs are the same as LO-pairs. For instance, a number of contexts have been identified in which the “survival-pair” and “LO-pair” concepts coincide: cf. [6, Theorem 2.7, Remark 2.8(a),(b),(c),(e)]. Another similarity of behavior arises from the fact that “survival-pair” plays the same role as “LO-pair” in sharpenings of the “Folklore Theorem” characterizing integrality [6, Theorem 2.1]. Similar phenomena occur in a “nearly integral” context [7, Theorem 2.14, Proposition 2.16] and in a result concerning complete integral closure [8, Proposition 2.1]. In our deepest result, Theorem 2.2, we give the underlying reason: survival-pairs are the same as LO-pairs.

In a number of contexts mentioned above (such as in [6, Theorem 2.7, Remark 2.8(a),(b),(c)]), the “survival-pair” condition is strong enough to imply integrality. Given the above-mentioned work in [5, Theorem, p. 38] and [1] on algebraic aspects of INC-pairs, it thus seems natural to pursue deeper connections between the “survival-pair” and “integrality” concepts. In this regard, Corollary 2.3 gives a new way in which the “survival-pair” concept can be used to characterize an integral ring extension. Moreover, Proposition 2.4 identifies a
sense in which arbitrary survival-pairs (that is, LO-pairs) are “closer” to being integral than are arbitrary INC-pairs.

Moving beyond the context of ring extensions to that of ring homomorphisms, recall for motivation that integrality and LO are universal properties (in the sense of [11], namely, being preserved by arbitrary base change) and that “universally GU” is equivalent to integrality. It seems natural to ask if/how these facts relate to the natural generalizations of the “survival” and “survival-pair” concepts to ring homomorphisms. The answers are given in Proposition 2.6(c) and Theorem 2.7: a ring homomorphism is integral (resp., satisfies LO) if and only if it is universally a survival-pair homomorphism (resp., universally a survival homomorphism). While these results frankly do not seem as deep to us as Theorem 2.2, they are pleasantly complete, they serve to place the “survival”-related concepts into a more appropriate categorical setting, and we would hope that readers find them of potential use in other studies.

We next describe notational conventions. If \( A \) is a ring, then \( \text{Spec}(A) \) denotes the set of prime ideals of \( A \); \( U(A) \) denotes the set of units of \( A \); and \( J(A) \) denotes the Jacobson radical of \( A \). If \( f \) is a ring homomorphism, then \( \ker(f) \) denotes the kernel of \( f \). By an overring of an integral domain \( R \), we mean any ring contained between \( R \) and its quotient field. Finally \( \subset \) denotes proper containment, and \( X \) denotes an indeterminate over the appropriate coefficient ring. Any unexplained material is standard, as in [12].

2 RESULTS

We begin by collecting some facts that will be useful in the proof of our main result.

**Lemma 2.1.** Let \((R,T)\) be a survival-pair. Then:

(a) If \( J \) is an ideal of \( T \) and \( I := J \cap R \), then \((R/I,T/J)\) is a survival-pair.

(b) If, in addition, \( R \) is integrally closed in \( T \), then \( U(R) = U(T) \).

**Proof:**

(a) We show that \( A \subseteq T/J \) is a survival extension for each ring \( A \) between \( R/I \) and \( T/J \). Identifying \( R/I = (R + J)/J \), we see that \( A = S/J \) for some ring \( S \) between \( R + J \) and \( T \). If survival fails for \( A \subseteq T/J \), then \( 1 \in Q(T/J) \) for some \( Q \in \text{Spec}(A) \). Write \( Q = q/J \) with \( q \in \text{Spec}(S) \). Then \( 1 \in qT + J = qT \) and \( R \subseteq T \) is a survival extension, contradicting that \( S \subseteq T \) is a survival extension.

(b) Deny. Choose \( v \in U(T) \setminus R \). As \( 1 \in vT = vR[v]T \) and \( R[v] \subseteq T \) is a survival extension, it follows that \( 1 \in vR[v] \), whence \( v^{-1} \in R[v] \) and \( v^{-1} \) is integral over \( R \) [12, Theorem 15]. Since \( v^{-1} \in T \) and \( R \) is integrally closed in \( T \), we now have that \( v^{-1} \in R \). As \( 1 = v^{-1}v \in Rv^{-1}T \) and \( R \subseteq T \) is a survival extension, \( Rv^{-1} = R \); that is, \( v^{-1} \in U(R) \). Then \( v = (v^{-1})^{-1} \in R \), the desired contradiction. \( \diamond \)

We next present our main result.

**Theorem 2.2.** Let \( R \subseteq T \) be rings. Then the following conditions are equivalent:

1. \( A \subseteq A[u] \) is a survival extension for all rings \( A \) such that \( R \subseteq A \subseteq T \) and all elements \( u \in T \);
(2) \( A \subseteq A[u] \) satisfies LO for all rings \( A \) such that \( R \subseteq A \subseteq T \) and all elements \( u \in T \); 

(3) \( A \subseteq A[u] \) satisfies GU for all rings \( A \) such that \( R \subseteq A \subseteq T \) and all elements \( u \in T \); 

(4) \((R,T)\) is a survival-pair; 

(5) \((R,T)\) is an LO-pair; 

(6) \((R,T)\) is a GU-pair.

**Proof:** It was shown in [6, Corollary 3.2] that conditions (5) and (6) are equivalent. Moreover, (6) implies (3) trivially, by the definition of \( \omega \)-pair. Also, (3) implies (2), since GU implies LO for extensions; and (2) implies (1) since LO implies survival for extensions. It remains to prove that (1) implies (4) and that (4) implies (5).

(1)\(\implies\)(4): We give an indirect proof in the spirit of the proof of [6, Proposition 2.10]. Suppose that (1) holds but (4) fails. Then \( A \subseteq B \) is not a survival extension, for some rings \( R \subseteq A \subseteq B \subseteq T \). Pick \( P \in \text{Spec}(A) \) such that \( 1 \in PB \). Now consider the set 

\[ S := \{ D | D \text{ is a ring, } A \subseteq D \subseteq B, \text{ and } P \text{ survives in } D \}, \]

partially ordered by inclusion. Evidently, \( S \neq \emptyset \), for \( A \in S \). If \( \{ E_a \} \) is a chain in \( S \), then \( \bigcup E_a \in S \). (Indeed, if \( 1 = \sum_{i=1}^n p_i e_i \) with \( p_i \in P, e_i \in E_{a_i} \), then \( 1 \in \mathcal{P} E_{a_n} \), contradicting that \( E_{a_n} \in S \).) Thus, Zorn's Lemma applies, producing a maximal element \( A_0 \) of \( S \). Since \( P \) does not survive in \( B \), we have that \( A_0 \neq B \). Choose \( u \in B \setminus A_0 \). As \( 1 \notin \mathcal{P} A_0 \), (1) ensures that \( \mathcal{P} A_0 \) survives in \( A_0[u] \); that is, \( 1 \notin \mathcal{P} A_0[u] \). Thus \( A_0[u] \in S \), contradicting that \( A_0 \) is maximal in \( S \).

(4)\(\implies\)(5): It suffices to prove that if \((R,T)\) is a survival-pair, then \( R \subseteq T \) satisfies LO. Deny. Fix \( \mathcal{P} \in \text{Spec}(R) \) such that no prime ideal of \( T \) lies over \( \mathcal{P} \). By [12, Theorem 10], \( \mathcal{P} \) contains a minimal prime ideal \( P_0 \) of \( R \). By [12, Exercise 1, p. 41], \( P_0 = Q_0 \cap R \) for some \( Q_0 \in \text{Spec}(T) \). Then, by Lemma 2.1, \((R/P_0, T/Q_0)\) inherits the property of being a survival-pair from \((R,T)\). We claim that no prime ideal of \( T/Q_0 \) lies over \( \mathcal{P}/P_0 \). If not, there exists \( W \in \text{Spec}(T) \) such that \( Q_0 \subseteq W \) and \((W/Q_0) \cap (R/P_0) = \mathcal{P}/P_0 \). More precisely, \((W/Q_0) \cap ((R + Q_0)/Q_0) = (\mathcal{P} + Q_0)/Q_0 \). A straightforward calculation reveals that \( W \cap R = \mathcal{P} \), contrary to the choice of \( \mathcal{P} \). (Indeed, it is clear that \( \mathcal{P} \subseteq W \cap R \). For the reverse inclusion, any \( w \in W \cap R \) can be written as \( w = p + q_0 \), for some \( p \in \mathcal{P}, q_0 \in Q_0 \). Then \( q_0 = w - p \in R + R = R \), whence \( q_0 \in Q_0 \cap R = P_0 \subseteq \mathcal{P} \), so that \( w \in \mathcal{P} + \mathcal{P} = \mathcal{P} \), as desired.) This proves the above claim. Therefore, by passing from \((R,T)\) to \((R/P_0, T/Q_0)\), we may assume, without loss of generality, that \( R \) and \( T \) are integral domains.

We next reduce to the case in which \( \mathcal{P} \) is not lain over by any prime ideal of any subring of \( T \) that properly contains \( R \). To this end, consider the set 

\[ \Gamma := \{ (D,Q) | D \text{ is a ring, } R \subseteq D \subseteq T, Q \in \text{Spec}(D), Q \bigcap D = \mathcal{P} \} \]
together with the partial order on \( \Gamma \) given by
\[
(D_1, Q_1) \leq (D_2, Q_2) \iff [D_1 \subseteq D_2 \text{ and } Q_2 \cap D_1 = Q_1].
\]
As in the proof of [6, Proposition 2.10], Zorn’s Lemma applies to \( \Gamma \), producing a maximal element \((D_3, Q_3)\) of \( \Gamma \). Replacing \((R, T, \mathcal{P})\) with \((D_3, T, Q_3)\) effects the promised reduction.

In view of the above reduction and the Lying-over Theorem (cf. [12, Theorem 44]) for integral extensions, we see easily that \( R \) is integrally closed in \( T \). Hence, by Lemma 2.1(b), \( U(R) = U(T) \). Next, since \( R \subseteq T \) does not satisfy LO, \( R \neq T \), and so we can fix \( u \in T \setminus R \). By the above reduction, no prime ideal of \( R[u] \) lies over \( \mathcal{P} \), and so we may replace \( T \) with \( R[u] \). Thus, without loss of generality, \( T = R[u] \). Moreover, \( u \) is algebraic over \( R \) (since \( PR[X] \in Spec(R[X]) \) and \( PR[X] \cap R = \mathcal{P} \)). It now follows from [12, Exercise 35, p. 44] that \( T \) is an overring of \( R \).

We next proceed to show that \( u^{-1} \in PR_F \). First, we claim that \( 1 \in PR_F[u] \).

If not, choose a prime ideal \( Q \) of \( PR_F[u] = R[u]_{R \setminus \mathcal{P}} = T_{R \setminus \mathcal{P}} \) such that \( PR_F[u] \subseteq Q \). Then, necessarily, \( Q \cap R = PR_F \), whence \( q := Q \cap T \in Spec(T) \) satisfies \( q \cap R = PR_F \cap R = \mathcal{P} \), contrary to the choice of \( \mathcal{P} \). This proves the above claim, namely, that \( 1 \in PR_F[u] \). In particular, \( u \) is a root of a polynomial in \( R_P[X] \) having a unit coefficient. Moreover, since \( R \) is integrally closed in \( T \), we see, as in the proof of [12, Theorem 51], that \( R_P \) is integrally closed in \( T_{R \setminus \mathcal{P}} \). Thus, by the proof of the \((u, u^{-1})\) Lemma [12, Theorem 67], either \( u \in R_P \) or \( u^{-1} \in R_P \). Notice that the ring \( E := R_P \cap T \) has the property that \( R \subseteq E \subseteq T \), \( q^* := PR_P \cap T \in Spec(E) \) and \( q^* \cap R = \mathcal{P} \). Accordingly, by an earlier reduction, \( E = R \). As \( u \in T \setminus R \), it follows that \( u \notin R_P \). Therefore, \( u^{-1} \in R_P \setminus U(R_P) = PR_P \), as promised. Fix a description \( u = ab^{-1} \), with \( a \in R \setminus \mathcal{P} \) and \( b \in \mathcal{P} \setminus \{0\} \).

Since \( R \) is integrally closed in \( T = R[u] = R[ab^{-1}] \), we have that \( R \) is also integrally closed in \( S := R[au] = R[a^2b^{-1}] = R[v] \), where \( v := 1 + a^2b^{-1} \).

Observe that the ideal \( I := Sb + Sb \) of \( S \) does not survive in \( T \), since \( 1 = v + b(-u^2) \in IT \). Consequently, as \((4)\) ensures that \( S \subseteq T \) is a survival extension, \( I = S \).

Thus
\[
(r_0 + r_1 v + \cdots + r_{n-1} v^{n-1})v + (r_0^* + r_1^* v + \cdots + r_n^* v^n) \cdot b = 1
\]
for some elements \( r_i, r_i^* \in R \). Defining
\[
t := (r_0 + br_0^*) + v(r_1 + br_2^*) + \cdots + v^{n-1}(r_{n-1} + br_n^*) \in S,
\]
we may rewrite the above equation as \( tv + r_0^* b = 1 \). Next, observe that \( v \neq 0 \), for \( r_0^* b \in Rb \subseteq \mathcal{P} \). Thus, \( z := tv = 1 - r_0^* b \in R \) satisfies \( t = vz^{-1} \in R[v^{-1}] \).

Since \( t \in S = R[v] \), we can now conclude that \( t \) is integral over \( R \). (The point is that, being in \( R[v] \cap R[v^{-1}] \), \( t \) must be a member of each valuation overring of \( R \); one then applies [12, Theorem 57] to the integral closure of \( R \).) Therefore, \( t \in R \), as \( R \) is integrally closed in \( T \).
In fact, \( t \in \mathcal{P} \). To see this, use the fact that \( \mathcal{P} \) is a prime ideal of \( R \), noticing that
\[
t(b + a^2) = b[t(1 + a^2b^{-1})] = b[tv] = b[1 - r_0b] \in bR \subseteq \mathcal{P},
\]
while \( b + a^2 \in R \setminus \mathcal{P} \). Hence \( 1 = tv + br_0 \in \mathcal{P}S \), contradicting the fact that (4) ensures that \( R \subseteq S \) is a survival extension. The proof is complete. \( \diamondsuit \)

We next infer an enhancement of [6, Corollary 3.6].

**Corollary 2.3.** Let \( R \subseteq T \) be rings. Then \( T \) is integral over \( R \) if and only if \((R[X], T[X])\) is a survival-pair.  

**Proof:** The “only if” assertion is immediate from the Going-up Theorem for integral extensions (cf. [12, Theorem 44]) and the fact that GU implies survival for extensions. The “if” assertion follows from the implication (4) implies (6) in Theorem 2.2 and the fact [4, Lemma, p. 160] that \( T \) is integral over \( R \) provided that \( R[X] \subseteq T[X] \) satisfies GU. \( \diamondsuit \)

As a companion for [6, Proposition 4.5], we next identify an instance of integrality within an arbitrary survival-pair. Following [3], we say that distinct rings \( A \subset B \) are adjacent provided that no ring \( D \) satisfies \( A \subset D \subset B \).

**Proposition 2.4.** Let \( R \subset T \) be distinct rings such that \((R, T)\) is a survival-pair. Then there exist adjacent rings \( A \subset B \) such that \( R \subseteq A \subset B \subseteq T \) and \( B \) is integral over \( A \).

**Proof:** The following proof is inspired by that of [12, Theorem 11]. Using Zorn’s Lemma, pick a maximal chain \( \{S_\alpha\} \) of rings contained between \( R \) and \( T \). Fix \( t \in T \setminus R \). Put \( A := \bigcup\{S_\alpha| t \notin S_\alpha\} \) and \( B := \bigcap\{S_\alpha| t \in S_\alpha\} \). If \( t \notin S_\alpha \) and \( t \in S_\beta \), then \( S_\beta \nsubseteq S_\alpha \), whence \( S_\alpha \subseteq S_\beta \). It follows that \( A \subseteq B \). In fact, \( A \subset B \), since \( t \in B \setminus A \). Moreover, \( A \) and \( B \) are adjacent, for if \( D \) were a ring contained properly between \( A \) and \( B \), then \( \{S_\alpha\} \cup \{D\} \) would be a chain contradicting the maximality of \( \{S_\alpha\} \). Finally, [3, (2.5.3)] ensures that \( A \subseteq B \), being an adjacent survival extension, is necessarily integral. \( \diamondsuit \)

It is interesting to note that INC-pairs do not satisfy an analogue of Proposition 2.4. Indeed, if \( R \) is a valuation domain of Krull dimension 1 and \( T \) is the quotient field of \( R \), then \((R, T)\) is an INC-pair such that \( R \subset T \) is a non-integral extension of adjacent rings.

We pause for some comments about Lemma 2.1.

**Remark 2.5.** (a) In view of the equivalence of conditions (4) and (5) in Theorem 2.2, we see that Lemma 2.1(a) is a generalization of [6, Lemma 3.1(b)] to the case in which the ideal \( J \) is not necessarily prime.

(b) The hypothesis that \( R \) is integrally closed in \( T \) cannot be deleted from the statement of Lemma 2.1(b). To see this, it suffices to consider any integral extension \( R \subseteq T \) such that \( U(T) \not\subseteq R \), for instance the extension \( \mathbb{Z} \subseteq \mathbb{Z}[i] \).
As is well known, notions such as LO and GU can be generalized from the “extension” context of [12, p. 28] to that of homomorphisms. While much carries over to the “homomorphism” context, care is needed as, for instance, [12, Theorem 42] does not generalize. (The point is that GU does not imply LO for homomorphisms. Consider, for example, the canonical surjection \( R \rightarrow R/M \), where \( R \) is a ring of Krull dimension 0 which is not a field, such as \( R = \mathbb{Q} \times \mathbb{C} \), and \( M \) is a prime ideal of \( R \).)

Undaunted, we say that a ring homomorphism \( f : R \rightarrow T \) is a survival homomorphism in case \( IT \neq T \) (that is, \( f(I)T \neq T \)) for all proper ideals \( I \) of \( R \). This notion is characterized in terms of “survival extension” in Proposition 2.6(a); its relation to LO (for homomorphisms) is pursued in Proposition 2.6(b) and Theorem 2.7. A homomorphism-theoretic analogue of Corollary 2.3 is given in Proposition 2.6(c), which depends on the following definition. A ring homomorphism \( f : R \rightarrow T \) is a survival-pair homomorphism in case \( A \subseteq B \) is a survival extension for all \( R \)-subalgebras \( A \subseteq B \) of \( T \).

**Proposition 2.6.** Let \( f : R \rightarrow T \) be a ring homomorphism. Then:

(a) \( f \) is a survival homomorphism if and only if \( f(R) \subseteq T \) is a survival extension such that \( \ker(f) \subseteq J(R) \).

(b) If \( f \) satisfies LO, then \( f \) is a survival homomorphism.

(c) \( f \) is integral if and only if \( f \) is universally a survival-pair homomorphism (in the sense that, for each ring homomorphism \( R \rightarrow S \), the induced homomorphism \( S \rightarrow S \otimes_R T \) is a survival-pair homomorphism).

**Proof:** (a) Put \( K := \ker(f) \). Suppose first that \( f \) is a survival homomorphism. Consider a proper ideal \( I \) of \( f(R) \). Using the identification \( f(R) = R/K \), we have \( I = I/K \) for some proper ideal \( I \) of \( R \) such that \( K \subseteq I \). Observe that \( IT = IT \). The hypothesis on \( f \) ensures that \( IT \neq T \), whence \( IT \neq T \); that is, \( f(R) \subseteq T \) is a survival extension. Now, suppose that \( K \not\subseteq M \) for some maximal ideal \( M \) of \( R \). Then \( K + M = R \), so that \( a + b = 1_R \) for some \( a \in K, b \in M \). Then, in \( T \), \( 1 = f(1_R) = f(a) + f(b) = 0 + b \cdot 1 = b \cdot 1 \in MT \), contradicting the hypothesis that \( MT \neq T \). Accordingly, no such \( M \) exists; that is, \( K \subseteq J(R) \). This completes the proof of the “only if” assertion.

Conversely, suppose that \( f(R) \subseteq T \) is a survival extension such that \( K \subseteq J(R) \). Our task is to show that if an ideal \( I \) of \( R \) satisfies \( IT = T \), then \( I = R \). Once again using the identification \( f(R) = R/K \), we have \( T = IT = ((I+K)/K)T \). As \( f(R) \subseteq T \) is a survival extension, it follows that \( (I+K)/K = f(R) = R/K \), whence \( I+K = R \). Write \( c+d = 1 \), for some \( c \in I, d \in K \). Then \( I \) contains \( c = 1 - d \in 1 + K \subseteq 1 + J(R) \subseteq U(R) \), whence \( I = R \).

(b) Deny. Then \( IT = T \) for some proper ideal \( I \) of \( R \). Choose \( \mathcal{P} \in \text{Spec}(R) \) such that \( I \subseteq \mathcal{P} \). As \( f \) satisfies LO, there exists \( Q \in \text{Spec}(T) \) such that \( f^{-1}(Q) = \mathcal{P} \). Then \( f(\mathcal{P}) \subset Q \), whence \( T = IT \subseteq \mathcal{P}T = f(\mathcal{P})T \ subset Q \), the desired contradiction.

(c) Suppose first that \( f \) is integral. We must prove that \( A \subseteq B \) is a survival extension whenever \( A \subseteq B \) are \( S \)-algebras of \( S \otimes_R T \) arising from a ring homomorphism \( R \rightarrow S \). It suffices to show that \( B \) is integral over \( A \), and this, in turn, follows since \( S \rightarrow S \otimes_R T \) inherits integrality from \( f \).
Conversely, suppose that $f$ is universally a survival-pair homomorphism. Using $S := R[X]$, we conclude that $R[X] \to R[X] \otimes_R T = T[X]$ is a survival-pair homomorphism. Thus, $A \subseteq B$ is a survival extension for all $R[X]$-subalgebras $A \subseteq B$ of $T[X]$. In other words, $(f(R)[X], T[X])$ is a survival-pair. Therefore, by Corollary 2.3, $T$ is integral over $f(R)$; equivalently, $f$ is integral, as asserted. ◦

**Theorem 2.7.** Let $f : R \to T$ be a ring homomorphism. Then $f$ satisfies LO if and only if $f$ is universally a survival homomorphism (in the sense that, for each ring homomorphism $R \to S$, the induced homomorphism $S \to S \otimes_R T$ is a survival homomorphism).

**Proof:** It is well known that LO is a universal property. (This can be shown by, for instance, applying [11, Corollaire 3.2.7.1(i), p. 235].) Thus, if $R \to S$ is any ring homomorphism and $f$ satisfies LO, then so does the induced map $S \to S \otimes_R T$. An application of Proposition 2.6(b) completes the proof of the "only if" assertion.

Conversely, suppose that $f$ is universally a survival homomorphism. Put $K := \ker(f)$. Suppose the assertion fails. Choose $\mathcal{P} \in \text{Spec}(R)$ such that no $Q \in \text{Spec}(T)$ satisfies $f^{-1}(Q) = \mathcal{P}$. Consider $I := f^{-1}(\mathcal{P} T)$, an ideal of $R$ such that $K \subseteq I$ and $\mathcal{P} \subseteq I$. The following analysis involves two cases.

In the first case, $I = \mathcal{P}$. We claim that $IT$ is disjoint from the multiplicatively closed subset $f(R \setminus \mathcal{P})$ of $T$. If not, choose $b \in R \setminus \mathcal{P}$ such that $f(b) \in IT$. Then, by a standard homomorphism theorem,

$$b \in f^{-1}(IT) = f^{-1}(f(I)T) = f^{-1}(\mathcal{P}T) = I,$$

so that $b \in I \setminus \mathcal{P} = \emptyset$, a contradiction. This proves the above claim. Therefore, by Zorn’s Lemma (cf. [12, p. 2]), there exists $Q \in \text{Spec}(T)$ such that $IT \subseteq Q$ and $Q \cap f(R \setminus \mathcal{P}) = \emptyset$. Observe that $\mathcal{P} = I = f^{-1}(f(I)) \subseteq f^{-1}(IT) \subseteq f^{-1}(Q)$. On the other hand, the fact that $Q \cap f(R \setminus \mathcal{P}) = \emptyset$ readily entails that $f^{-1}(Q) \subseteq \mathcal{P}$. Thus, $f^{-1}(Q) = \mathcal{P}$, the desired contradiction.

In the remaining case, $I \neq \mathcal{P}$. Fix $r \in I \setminus \mathcal{P}$. Applying the hypothesis with $S := R_r$, we have that $R_r \to R_r \otimes_R T = T_r$ is a survival homomorphism. (Of course, $T_r$ is $R_r$-algebra isomorphic to $T_f(r)$.) In particular, $1 \notin (\mathcal{P} R_r) T_r = \mathcal{P} T_r$. However, in $T_r$,

$$1 = rr^{-1} \in IT_r = f^{-1}(PT) T_r \subseteq (PT) T_r = \mathcal{P} T_r,$$

the desired contradiction. The proof is complete. ◦

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