AP-DOMAINS AND UNIQUE FACTORIZATION

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Abstract. In this paper we generalize the standard notion of "unique factorization domains" (UFDs) to the the nonatomic situation. The main result of this paper is that, in contrast to the atomic situation, the assumption that every irreducible is prime (AP) and the notion that every (atomic) nonzero nonunit can be factored uniquely into irreducible elements are distinct notions.

1. Introduction

A number of authors have studied factorization properties of commutative rings with identity. This situation has been generalized with some success to the situation of rings with non-trivial zero-divisors [1]. One situation that has been overlooked for the most part is the nonatomic situation. Although the extreme case ("extreme" meaning the "antimatter" situation of no atoms whatsoever) has been explored by Dobbs, Mullins, and the first author [3], in practice most domains are neither atomic nor antimatter. Indeed, if R is any antimatter domain (that is not a field), then its polynomial ring $R[x]$ is not atomic, but does contain atoms (e.g., the indeterminate x).

The standard definitions of "factorization domains" (e.g., UFDs, HFDs, BFDs, etc.) always include the assumption that the domain is atomic (i.e., every nonzero, nonunit of the domain can be written as a product of irreducible elements or atoms). It is natural (and perhaps imperative) to consider the implications to the theory when this assumption is dropped. For example, one might declare more generally that a domain, R, is an *unrestricted unique factorization domain* (U-UFD) if every element that *can* be factored uniquely into irreducible elements has unique factorization. More specifically, if

$$\alpha_1\alpha_2\cdots\alpha_n = \beta_1\beta_2\cdots\beta_m$$

with each α_i, β_j irreducible in R, then $n = m$ and there is an element $\sigma \in S_n$ such that α_i is an associate of $\beta_{\sigma(i)}$ for all $1 \leq i \leq n$.

Of course, in the atomic case, this definition collapses to the standard notion of UFD.

There are a number of equivalent characterizations of UFD and for the purposes of factorization, some are more streamlined than others. For example, it is known that the (generally weaker) notion of "irreducible element" is equivalent to the notion of "prime element." More generally, UFDs are atomic domains where every irreducible element is prime (uniqueness "comes for free" in prime factorizations). The general domain type where the notions of irreducible and prime coincide is

1991 *Mathematics Subject Classification.* Primary: 13G05; Secondary: 13F15.
Key words and phrases. Factorization, AP-domains, U-UFDs.

called an *AP domain* (AP for atoms prime). These domains are well-known and have been studied in [2] among others.

As has been pointed out, the notions of AP and U-UFD are equivalent in the atomic case. For the more general theory, it is imperative to determine if these properties are equivalent. In particular, it is clear that any AP domain necessarily has the U-UFD property. Since AP-domains have a number of nice ring-theoretic properties, we would like to know if all U-UFDs are AP-domains (and since the theories coincide in the atomic case, we would like to know if they coincide universally).

The bulk of this paper is devoted to showing that the notions of AP and U-UFD are distinct ones. The construction used to distinguish the properties involves a large direct limit construction, and from a practical point of view, the properties are somewhat difficult to distinguish (all examples in the literature of domains that have the U-UFD property are, in fact, AP domains).

Before proceeding, we remark that in this paper, R will be an integral domain with quotient field K. The units and irreducible elements of R will be denoted by $U(R)$ and $Irr(R)$, respectively.

2. Comparison of the U-UFD and AP notions

We first make/recall some preliminary definitions.

Definition 2.1. *Let R be an integral domain.*
1) We say that R is an unrestricted UFD (U-UFD) if every element that can be factored into irreducibles has a unique (up to units) factorization into irreducibles.
2) We say that R is an AP-domain if every irreducible element of R is prime.

From one point of view, the known class of AP-domains would be a good choice for a class of unrestricted unique factorization domains since it is clear that any atomic AP-domain is a UFD. It is also easy to see that any AP-domain is automatically a U-UFD (and we will show this presently). These observations lead to the natural question as to whether the concept of U-UFD and the known concept of AP are equivalent. Many of the standard examples of U-UFDs (e.g., a valuation domain has at most one irreducible, and when an irreducible is present, it is prime) that one would naturally investigate are, in fact, AP-domains. As we know of no "simple" example, this section will be devoted to showing that the notions of U-UFD and AP are, in fact, distinct.

As was noted in the introduction, the concepts of AP-domain and U-UFD are similar. For the sake of completeness we record the following lemma.

Lemma 2.2. *If R is an AP-domain, then R is a U-UFD.*

Proof: Suppose $r \in R$ is a nonzero nonunit that can be factored into irreducible elements. We write:

$$r = \pi_1 \pi_2 \cdots \pi_n$$

with each $\pi_i, 1 \leq i \leq n$ irreducible in R.

Since R is an AP-domain, each $\pi_i, 1 \leq i \leq n$ is a prime element of R. It is well-known [4] that any factorization into primes is unique. This concludes the proof. $\diamondsuit$

With this in hand, the question as to whether the concepts of AP and U-UFD are equivalent boils down to the question of the existence of a U-UFD with nonprime irreducibles (and indeed such domains exist). Our strategy will be to construct a domain with a unique non-prime irreducible and apply the following lemma.

Lemma 2.3. *Let R be a domain with a unique (up to associates) irreducible element, π. One of the following conditions must be satisfied.*
1) If π is prime then R is AP (and if R is atomic then R is a Noetherian valuation domain).
2) If π is not prime then R is a U-UFD that is not AP (and hence R is not atomic).

Proof: For the first statement, the fact that R is AP is clear as there is a unique irreducible which is prime. Also note that if R is atomic then every nonzero element of R is of the form $u\pi^n$ with $u \in U(R)$ and $n \geq 0$. Hence R is a Noetherian valuation domain.

For the second statement, we suppose that $\pi \in R$ is a non-prime irreducible and we choose an element $x \in R$ such that x has a factorization into irreducibles. As π is the only irreducible in R, we must have that $x = u\pi^n$ for some $u \in U(R)$ and $n \geq 1$. For uniqueness (and hence the U-UFD property), we consider the two factorizations:

$$x = u\pi^n = v\pi^m$$

with $u, v \in U(R)$, $n, m \geq 1$.

Given these factorizations, the only way to deny uniqueness is for (without loss of generality) $n > m$. But if this is the case, then we can divide both sides of the above equation by π^m to obtain:

$$u\pi^{n-m} = v.$$

Since $n-m > 0$ this implies that a positive power of π is a unit. But π is irreducible and we have our desired contradiction. $\diamondsuit$

With the previous lemma in hand, we now endeavor to construct a domain that possesses a unique non-prime irreducible element. We will begin by considering a domain, R, that possesses a non-prime irreducible element. We choose a non-prime irreducible $\pi \in R$. From this starting point we will construct a tower of domains ($\{R_i\}_{i=0}^{\infty}$ with $R_i \subseteq R_{i+1}$) containing R such that at each stage π remains prime in R_i and no other irreducible in R_i remains irreducible in R_{i+1}. With this tower constructed, we will consider $\bigcup(R_i)$.

To help implement this construction we begin with a few lemmas. We would like to note at this juncture that a few of the lemmas introduced here may be found implicitly in an interesting paper of Roitman ([5]). The goal in [5] was to construct an atomic domain, R, with the property that $R[x]$ was not atomic. The goal of this paper is decidedly different. For this reason (and for the sake of self-containment) we offer brief proofs of the necessary lemmas.

Lemma 2.4. *Let R be a domain with quotient field K and let $S = \{s_i\}_{i\in\Lambda}$ denote a family of elements of R and $\{x_i\}_{i\in\Lambda}$ indeterminates over R. Then $K \bigcap R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Lambda} = R$.*

Proof: We first investigate the one-variable case. With notation as in the statement of the lemma, we consider the intersection

$$K\bigcap R[x][\frac{s}{x}]$$

and first note that this domain clearly contains R. For the other containment, we let $\lambda\in K$ and assume that $\lambda\in R[x][\frac{s}{x}]$. As $\lambda\in R[x][\frac{s}{x}]$, we can write

$$\lambda=r(x)+\frac{r_1s}{x}+\frac{r_2s^2}{x^2}+\cdots+\frac{r_ks^k}{x^k}$$

with $r_i\in R$ for $1\leq i\leq k$ and $r(x)\in R[x]$.

We reduce the problem to the polynomial case by multiplying both sides of the above equation by x^k obtaining

$$\lambda x^k=x^kr(x)+r_1sx^{k-1}+\cdots+r_ks^k.$$

Equating coefficients, we see that $r_i=0$ for all $1\leq i\leq k$ and that $r(x)=\lambda\in R[x]\bigcap K=R$. This establishes the one-variable case.

Inductively, we assume that

$$K\bigcap R[x_1,\cdots,x_m][\frac{s_1}{x_1},\cdots,\frac{s_m}{x_m}]=R.$$

Letting $T:=R[x_1,\cdots,x_m][\frac{s_1}{x_1},\cdots,\frac{s_m}{x_m}]$, we note that the inductive hypothesis gives that

$$F\bigcap T[x_{m+1}][\frac{s_{m+1}}{x_{m+1}}]=T$$

where $F=K(x_1,\cdots,x_m)$ is the quotient field of T.

Noting that $K\subseteq F$, we intersect the above with K and obtain

$$K\bigcap F\bigcap T[x_{m+1}][\frac{s_{m+1}}{x_{m+1}}]=K\bigcap T[x_{m+1}][\frac{s_{m+1}}{x_{m+1}}]=K\bigcap T$$

and by induction $K\bigcap T=R$. Hence we have established that

$$K\bigcap T[x_{m+1}][\frac{s_{m+1}}{x_{m+1}}]=K\bigcap R[x_1,\cdots,x_m,x_{m+1}][\frac{s_1}{x_1},\cdots,\frac{s_m}{x_m},\frac{s_{m+1}}{x_{m+1}}]=R.$$

The general case (for an arbitrary index set) now follows in a similar fashion. If α is any element of $K\bigcap R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Lambda}$, then α can be written as a finite sum consisting of only finitely many of the indeterminates $x_i, i\in\Lambda$. Hence α is in R by the above. $\diamondsuit$

Lemma 2.5. *Let R be a domain and let $S=\{s_i\}_{i\in\Lambda}$ denote a family of nonunit elements of R and $\{x_i\}_{i\in\Lambda}$ indeterminates over R. Then $U(R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Lambda})=U(R)$.*

Proof: We first verify the one-variable case. We choose a unit, u, in the domain $R[x][\frac{s}{x}]$.

$$u=r(x)+\frac{r_1s}{x}+\frac{r_2s^2}{x^2}+\cdots+\frac{r_ks^k}{x^k}$$

with $r_i\in R$ for $1\leq i\leq k$ and $r(x)\in R[x]$. Since u is a unit we can find a v with $uv=1$. In particular,

$$(u)(v) = (r(x) + \frac{r_1 s}{x} + \frac{r_2 s^2}{x^2} + \cdots + \frac{r_k s^k}{x^k})(\overline{r}(x) + \frac{\overline{r}_1 s}{x} + \frac{\overline{r}_2 s^2}{x^2} + \cdots + \frac{\overline{r}_k s^k}{x^k}) = 1$$

with notation as above (we have chosen "k" for the terminal denominator exponent for v without loss of generality).

Multiplying the above equation by x^{2k}, we obtain

$$(x^k r(x) + x^{k-1} r_1 s + x^{k-2} r_2 s^2 + \cdots + r_k s^k)(x^k \overline{r}(x) + x^{k-1}\overline{r}_1 s + x^{k-2}\overline{r}_2 s^2 + \cdots + \overline{r}_k s^k) = x^{2k}.$$

At this juncture we consider a couple of cases. First, we consider the case where neither $r(x)$ nor $\overline{r}(x)$ is 0. In this case, we note that both $r(x)$ and $\overline{r}(x)$ are constants and, in fact, inverses of one another. Multiplying out the above equation gives

$$x^{2k} + s(p(x)) = x^{2k}$$

where $s(p(x))$ is the polynomial consisting of the "cross terms." Inspection of the above shows that $s(p(x)) = 0$, and it is easy to see (as monomials form a saturated multiplicative set) that u is of the form

$$u = r(x)$$

but as we have established that u is of degree 0 and invertible, $u \in U(R)$.

The next case to consider is where (at least one of) $r(x)$ or $\overline{r}(x)$ is 0. But inspection of the second equation above shows that if either $r(x)$ or $\overline{r}(x)$ is 0, then the left side of the equation is in the ideal generated by s (a nonunit) and the right side (x^{2k}) is not. This establishes the one-variable case.

Inductively, we assume that

$$U(R[x_1, \cdots, x_m][\frac{s_1}{x_1}, \cdots, \frac{s_m}{x_m}]) = U(R)$$

and establish the result for $m+1$. Letting $T := R[x_1, \cdots, x_m][\frac{s_1}{x_1}, \cdots, \frac{s_m}{x_m}]$, we note that

$$U(T[x_{m+1}][\frac{s_{m+1}}{x_{m+1}}]) = U(T)$$

by induction. But as $U(T) = U(R)$ by induction as well, we have

$$U(R[x_1, \cdots, x_m, x_{m+1}][\frac{s_1}{x_1}, \cdots, \frac{s_m}{x_m}, \frac{s_{m+1}}{x_{m+1}}]) = U(R)$$

and the lemma is established for the finite case.

The general case is an easy observation as in the previous lemma. Indeed, if u is a unit in $R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Lambda}$ then as it is a finite sum, u can be thought of as an element of $R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Gamma}$ for some *finite* index set Γ (and hence $u \in U(R)$). This completes the proof.$\diamondsuit$

Lemma 2.6. *Let R be a ring, Λ be an indexing set and let $\xi_i, \pi \in Irr(R), i \in \Lambda$ all be pairwise non-associate and $\{x_i\}_{i\in\Lambda}$ indeterminates over R. Then the element π is irreducible as an element of $R[\{x_i\}][\{\frac{\xi_i}{x_i}\}]_{i\in\Lambda}$.*

Proof: We begin with the one-variable case ($R[x][\frac{\xi}{x}]$) by again noting that a typical element of the ring $R[x][\frac{\xi}{x}]$ can be written in the form:

$$p(x)+r_1\frac{\xi}{x}+r_2(\frac{\xi}{x})^2+\cdots+r_k(\frac{\xi}{x})^k$$

with $p(x)\in R[x]$ and $r_i\in R$ for $1\leq i\leq k$. Suppose that π can be factored in $R[x][\frac{\xi}{x}]$. We write

$$\pi=(p(x)+r_1\frac{\xi}{x}+r_2(\frac{\xi}{x})^2+\cdots+r_k(\frac{\xi}{x})^k)(q(x)+s_1\frac{\xi}{x}+s_2(\frac{\xi}{x})^2+\cdots+s_k(\frac{\xi}{x})^k)$$

with $p(x),q(x)\in R[x]$ and $r_i,s_i\in R$ for $1\leq i\leq k$ (we assume without loss of generality for the ease of notation that the two k's coincide). We multiply both sides of the above equation by x^{2k}

$$\pi x^{2k}=(x^kp(x)+r_1\xi x^{k-1}+\cdots+r_k\xi^k)(x^kq(x)+s_1\xi x^k+\cdots+s_k\xi^k)$$

to reduce to the ring $R[x]$. We first assume that neither $p(x)$ nor $q(x)$ is 0. In this case, a degree argument shows that both $p(x)\equiv p$ and $q(x)\equiv q$ must be constant (and nonzero). But as the left-hand side of the above equation is a monomial (and monomials form a saturated set in $R[x]$) our factorization reduces to $\pi x^{2k}=(x^kp)(x^kq)$, and hence $\pi=pq$. Without loss of generality we take p to be a unit and we are done with this case.

The last case to consider is the case where at least one of $p(x),q(x)$ is zero. We take $p(x)=0$ and note that in the displayed equation above, the product on the right-hand side is necessarily divisible by ξ which contradicts the fact that π and ξ are pairwise non-associate irreducible elements.

The general case follows easily in analogy to the previous lemmas. Indeed, if $\pi=ab$ with $a,b\in R[\{x_i\}][\{\frac{\xi_i}{x_i}\}]$, then each a,b can be written as a finite sum involving only finitely many of the introduced indeterminates (say $x_1,x_2,\cdots,x_n$, without loss of generality). By induction, one of the elements (say a) must be a unit in $R[x_1,\cdots,x_n][\frac{\xi_1}{x_1},\cdots,\frac{\xi_n}{x_n}]$ and hence a unit in $R[\{x_i\}][\{\frac{\xi_i}{x_i}\}]$. $\diamondsuit$

Lemma 2.7. *Let R be a ring and let $S=\{s_i\}_{i\in\Lambda}$ denote a family of nonzero nonunit elements of R and $\{x_i\}_{i\in\Lambda}$ indeterminates over R. In the extension ring $R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Lambda}$, the elements $\{s_i\}$ are reducible.*

Proof: It is evident that the element s_i factors as follows:

$$s_i=(x_i)(\frac{s_i}{x_i}).$$

The burden here is to show that the two factors x_i and $\frac{s_i}{x_i}$ are nonunits in the ring $R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i\in\Lambda}$. We will begin by showing the single variable case. That is, if $s\in R$ is irreducible, then in the ring $R[x][\frac{s}{x}]$ the elements x and $\frac{s}{x}$ are nonunits. Using the form of a typical element of $R[x][\frac{s}{x}]$ from the previous proof, we first assume that x is a unit in $R[x][\frac{s}{x}]$. In particular, we have

$$\frac{1}{x}=p(x)+r_1\frac{s}{x}+r_2(\frac{s}{x})^2+\cdots+r_k(\frac{s}{x})^k.$$

As before, we reduce to $R[x]$ by multiplying both sides of the above equation by x^k:

$$x^{k-1} = x^k p(x) + r_1 s x^{k-1} + r_2 s^2 x^{k-2} + \cdots + r_k s^k$$

But this equation immediately implies that $p(x) = 0$ which in turn shows that $x^{k-1} \in sR[x]$, which is our desired contradiction. The proof that $\frac{s}{x}$ is a nonunit is similar.

Since

$$R[x_1, \cdots, x_m, x_{m+1}][\frac{s_1}{x_1}, \cdots, \frac{s_m}{x_m}, \frac{s_{m+1}}{x_{m+1}}] = (R[x_1, \cdots, x_m][\frac{s_1}{x_1}, \cdots, \frac{s_m}{x_m}])[x_{m+1}][\frac{s_{m+1}}{x_{m+1}}]$$

the result for any finite number of indeterminates follows by induction.

This in turn gives the general case since if either x_i or $\frac{s_i}{x_i}$ is a unit in $R[\{x_i\}][\{\frac{s_i}{x_i}\}]_{i \in \Lambda}$ then the "inverse" will involve only a finite number of the (possibly infinite) family of indeterminates. This completes the proof. $\diamondsuit$

We now give the theorem that shows that there are U-UFDs in which not all atoms are prime. We remark that this construction can be generalized to non-AP U-UFDs which have more exotic properties than the one given. For our purposes, however, we found it expedient to construct a U-UFD with a unique non-prime irreducible. Such a domain is necessarily a U-UFD and by construction is not AP. We further remark that such a domain must necessarily be nonatomic.

Theorem 2.8. *There exist U-UFD rings that are not AP.*

Proof: We select a domain, R, with an irreducible element (say π) that is not prime. We then index the remaining irreducible elements (not associate to π) $\alpha^{(0)}_{i^{(0)}}$ and construct the ring

$$R_1 := R[x^{(0)}_{i^{(0)}}][\frac{\alpha^{(0)}_{i^{(0)}}}{x^{(0)}_{i^{(0)}}}]_{i^{(0)} \in \Lambda^{(0)}},$$

with each $x^{(0)}_{i^{(0)}}$ denoting an indeterminate over R. We note that as an element of R_1, π remains irreducible (by Lemma 2.6). Of course, there may be many "new" irreducible elements of R_1 (and, in fact, the techniques used in the proofs of Lemmas 2.6 and 2.7 can be used to show that if $r \in R$ is irreducible, then $\frac{r}{x}$ is an irreducible element of $R[x][\frac{r}{x}]$). We again set aside π and construct R_2 in a similar fashion. Generally, we construct R_{n+1} by indexing the irreducible elements not associate to π in R_n (say $\alpha^{(n)}_{i^{(n)}}$ with $i^{(n)} \in \Lambda^{(n)}$) and let

$$R_{n+1} := R[x^{(n)}_{i^{(n)}}][\frac{\alpha^{(n)}_{i^{(n)}}}{x^{(n)}_{i^{(n)}}}]_{i^{(n)} \in \Lambda^{(n)}},$$

again with each $x^{(n)}_{i^{(n)}}$ denoting an indeterminate over R.

At this stage we have constructed an increasing chain of domains:

$$R = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n \subseteq \cdots$$

and we utilize this chain to construct the directed union

$$T := \bigcup_{i=0}^{\infty} R_i.$$

It remains to show that T has the properties claimed; that is, T is a domain with a unique (up to associates) irreducible element.

Our first claim is that the element π is an irreducible element of T. To see this, we first note that π is irreducible in R_1 by Lemma 2.6. Hence by induction, π is irreducible in R_n for all $n \geq 0$. If $\pi = ab$ with $a, b \in T$ then there is a positive integer m such that both a and b are in R_m. By the above remark, one of a or b is a unit in R_m, hence a unit in T. This establishes the first claim.

Our second claim is that the element π is the *unique* (up to associates) irreducible in T. Indeed, if there is an irreducible element $\xi \in T$ that is not associate to π, then there is a positive integer m such that $\xi \in R_m$. By construction, $\xi = ab$ as an element of R_{m+1} with both a and b nonunits in R_{m+1}. Assume that in T, the element a (or b) becomes a unit. We note that Lemma 2.5 gives that

$$U(R) = U(R_1) = U(R_2) = \cdots = U(T).$$

Hence $a \in U(R_{m+1})$ which is our desired contradiction.

Lastly, we claim that π is a *non-prime* irreducible element of T. By our choice of π, there are elements $a, b \in R$ such that $\pi|(ab)$ but π divides neither a nor b (equivalently, neither $\frac{a}{\pi}$ nor $\frac{b}{\pi}$ is an element of R). Assume, without loss of generality, that $\frac{a}{\pi}$ is an element of T. Then $\frac{a}{\pi}$ is an element of R_m for some positive integer m. Hence, by Lemma 2.4 $\frac{a}{\pi} \in K \bigcap R_m = R_{m-1}$. By induction, $\frac{a}{\pi} \in R$ which is our contradiction.

So T possesses a unique (up to associates) non-prime irreducible element π. By Lemma 2.3, T is a U-UFD that is not AP. $\diamondsuit$

We find it interesting to note that our construction is necessarily nonatomic (since the notions of AP and U-UFD are the same in the atomic case). Although this fact might be intuitively obvious, we do not find it absolutely transparent.

It is also worth noting that this construction can be used to embed *any* domain which has a non-prime irreducible into a U-UFD that is not AP. We record this final result as a corollary.

Corollary 2.9. *Let R be a domain that has a non-prime irreducible π. Then R may be embedded in a U-UFD that is not AP. In particular, any atomic domain that is not a UFD can be embedded in a U-UFD that is not AP.*

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