THE PICARD GROUP OF A POLYNOMIAL RING

JIM COYKENDALL Department of Mathematics, Cornell University, Ithaca, New York 14853

1. INTRODUCTION AND NOTATION

In this paper R will denote an integral domain with quotient field K. The following is a list of some commonly used notation in this paper:

Z,Q,R -the rings of integers, rational numbers, and real numbers, respectively.
R$[x_1, ..., x_n]$ - the polynomial ring over R in n variables.
ι - the inclusion map.
$\varepsilon_{y=a}$ - the map from a polynomial ring given by the substitution y=a.
U(R) - the group of units of R.
f^* - the map induced by the functor Pic().
$f^\sharp$ - the map induced by the unit functor.
$\deg_x$ - the degree in the variable x of the given monomial.
NPicR$_n$ - the kernel of the map $\epsilon^*_{x_n=0}$ between PicR$[x_1, ..., x_n]$ and PicR$[x_1, ..., x_{n-1}]$.

We recall that an R-submodule I of K is called a fractional ideal of R if there is a nonzero a∈R such that aI⊆R. If I is a nonzero fractional ideal of R, we define I^{-1}={x∈K|xI⊆R}. Clearly II^{-1} ⊆R, and we say I is invertible if II^{-1}=R. The set Inv(R)={I|I is an invertible ideal of R} forms an abelian group under the usual multiplication of ideals; moreover, Prin(R)={xR|x∈K^*} is a subgroup of Inv(R). The quotient group Inv(R)/Prin(R) is defined to be the class group of R, denoted C(R). For a commutative ring R with identity, Pic(R) is defined to be the set of isomorphism classes of rank 1 finitely generated projective R-modules with [P]·[Q]=[P$\otimes_R$Q]. More generally, Pic is a functor from the category of commutative rings with identity to the category of abelian groups. In the case of domains, however, C(R) and Pic(R) are equivalent notions, so here we will use them interchangeably. A good reference for this background material is either [3] or [9]. For more information on Picard groups of polynomial rings see [2], [4], [5], [7], [10].

Results similar to some of the ones in this paper were done by B. Dayton and C. Weibel in [6]. Their paper is an excellent reference for a more "modern" approach to this material. Although the results achieved there are sharper in a certain sense, they are achieved in a more restricted environment and with very different techniques than we wished to use here. We would like to thank Dr. Weibel for making us aware of their results.

2. SEMINORMALITY AND THE SPLITTING OF PicR$[x_1, ..., x_n]$

Consider the following sequence of ring homomorphisms:

$$\mathrm{R} \overset{\iota}{\hookrightarrow} \mathrm{R[x]} \xrightarrow{\varepsilon_{x=0}} \mathrm{R}.$$

Clearly $\varepsilon\iota = 1_R$ and as Pic is a functor, this implies that $\varepsilon^*\iota^* = 1_{\mathrm{Pic(R)}}$. So we have:

$$\mathrm{Pic(R)} \xrightarrow{\iota^*} \mathrm{PicR[x]} \xrightarrow{\varepsilon^*_{x=0}} \mathrm{Pic(R)}.$$

Therefore we have the well-known result:

$$\mathrm{PicR[x]} \cong \mathrm{Pic(R)} \oplus \mathrm{NPicR}_1.$$

In general we have:

$$\mathrm{R}[x_1, ..., x_n] \overset{\iota}{\hookrightarrow} \mathrm{R}[x_1, ..., x_n, x_{n+1}] \xrightarrow{\varepsilon_{x_{n+1}=0}} \mathrm{R}[x_1, ..., x_n],$$

and as above we obtain:

$$\mathrm{PicR}[x_1, ..., x_{n+1}] \cong \mathrm{PicR}[x_1, ..., x_n] \oplus \mathrm{NPicR}_{n+1},$$

and inductively:

$$\mathrm{PicR}[x_1, ..., x_{n+1}] \cong \mathrm{PicR} \oplus \mathrm{NPicR}_1 \oplus ... \oplus \mathrm{NPicR}_{n+1}.$$

At this stage we would like to present an example to demonstrate that NPicR_n is not zero in general, but first we need some machinery.

DEFINITION 2.1. Let R be an integral domain with quotient field K. If a$\in$K and $a^2, a^3 \in R \Rightarrow a \in R$, then we say R is seminormal. We will say that a is nonseminormal with respect to R if $a \in K\backslash R$, but $a^2, a^3 \in R$.

THEOREM 2.2. $\mathrm{PicR} \cong \mathrm{PicR}[x_1, ..., x_n] \Longleftrightarrow$ R is seminormal.

Proof: See [7].

EXAMPLE 2.3. Let $R = k[s^2, s^3]$ with k a field and s an indeterminate. Clearly R is not seminormal. Let $I = (1+sx, s^2x^2)$ and $J = (1-sx, s^2x^2)$. It is easy to check that $J = I^{-1}$ and that $0 \neq [I] \in \ker(\mathrm{PicR[x]} \xrightarrow{\varepsilon^*_{x=0}} \mathrm{PicR})$. As R is seminormal$\Longleftrightarrow$R[x] is seminormal (see [5] or Theorem 2.2 above), we can see that $\mathrm{NPicR}_n \neq 0$ for all n>0.

EXAMPLE 2.4. Another example of a nonseminormal integral domain is the set of all analytic functions on the real line such that $f'(0)=0$. To see this merely think in terms of (convergent) power series.

EXAMPLE 2.5. Consider the ring R=$\mathbf{Z}$+$\mathbf{Z}$x+x$^2\mathbf{Q}$[x]. It is easy to see that R is not seminormal, but this example is distinguished from 2.3 by the fact that the ideal P=$(1+\frac{x}{2}t,\frac{x^2}{4}t)$, the analog of I above, is of finite order in NPicR$_1$ (see proof of Theorem 5.1).

3. THE CHARACTERIZATION OF NPicR$_n$

DEFINITION 3.1. A commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} A & \xrightarrow{g_1} & B \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ C & \xrightarrow{f_0} & D \end{array}$$

is called Cartesian if for every pair (b,c) in B×C such that $f_1(b)=f_0(c)$, there is a unique a∈A such that $g_1(a)=b$ and $g_0(a)=c$.

THEOREM 3.2. If the diagram

$$\begin{array}{ccc} A & \xrightarrow{g_1} & B \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ C & \xrightarrow{f_0} & D \end{array}$$

is a Cartesian square of commutative ring homomorphisms with either f_1 or f_0 surjective, then we have the following exact sequence:

$$0\longrightarrow U(A)\longrightarrow U(B)\oplus U(C)\longrightarrow U(D)\longrightarrow Pic(A)\longrightarrow Pic(B)\oplus Pic(C)\longrightarrow Pic(D).$$

Proof: See [3] or [9].

REMARK 3.3. The maps in the above exact sequence are the maps in the Mayer-Vietoris sequence.

Now we will use the above to explore some of the properties of the above-mentioned "NPics". Let A⊆B be subrings of an integral domain D, and let I be an additive subgroup of D such that:

1. jI⊆I for all j∈I
2. bI⊆I for all b∈B, and
3. B⋂I={0}.

Then we have that A+I⊆B+I are both subrings of D with common ideal I. Consider the sequence:

$$B\overset{\iota}{\hookrightarrow}B+I\xrightarrow{\varepsilon}B$$

with ε(b+i)=b. As $\varepsilon\iota=1_B$, we have Pic(B+I)≅Pic(B)⊕Ker(ε^*) and the analogous result for Pic(A+I).

THEOREM 3.4. The above situation induces a split exact sequence:

$$0\longrightarrow \mathrm{Pic(A+I)}\longrightarrow \mathrm{Pic(A)}\oplus\mathrm{Pic(B+I)}\longrightarrow \mathrm{Pic(B)}\longrightarrow 0.$$

Moreover, $\mathrm{Ker}(\varepsilon^*)\cong\mathrm{Ker}((\varepsilon|_{A+I})^*)$.

Proof: Consider the Cartesian square:

$$\begin{array}{ccc} \mathrm{A+I} & \hookrightarrow & \mathrm{B+I} \\ \downarrow & & \downarrow\varepsilon \\ A & \hookrightarrow & B. \end{array}$$

From Theorem 3.2 we have the following exact sequence:

$$0\longrightarrow \mathrm{U(A+I)}\longrightarrow \mathrm{U(A)}\oplus\mathrm{U(B+I)}\xrightarrow{\beta}\mathrm{U(B)}\longrightarrow$$
$$\longrightarrow \mathrm{Pic(A+I)}\longrightarrow \mathrm{Pic(A)}\oplus\mathrm{Pic(B+I)}\xrightarrow{\alpha}\mathrm{Pic(B)}.$$

For the first part of the theorem, it suffices to show that both α and β are surjective. β(x,y)=$\iota^\sharp$(x)$\varepsilon^\sharp$(y), where $\iota^\sharp$ and $\varepsilon^\sharp$ are induced by the unit functor. As $\varepsilon\iota$=1_B, we have that $\varepsilon^\sharp$ is surjective. Therefore, the restriction $\beta|_{U(B+I)}$ is onto, hence β is onto. An identical argument gives the surjectivity of α. The fact that $\varepsilon\iota$=1_B shows that it splits. Thus we have established the existence of the split exact sequence:

$$0\longrightarrow \mathrm{Pic(A+I)}\xrightarrow{f}\mathrm{Pic(A)}\oplus\mathrm{Pic(B+I)}\xrightarrow{g}\mathrm{Pic(B)}\longrightarrow 0.$$

Now we wish to show that $\mathrm{Ker}(\varepsilon^*)\cong\mathrm{Ker}((\varepsilon|_{A+I})^*)$. Consider $\mu\in\mathrm{Ker}((\varepsilon|_{A+I})^*)$. Let f($\mu$)=(0,b)$\in$Pic(A)$\oplus$Pic(B+I). Due to the exactness of the above sequence, g(0,b)=ι_A^*(0)+ε^*(b)=0. Therefore b$\in$Ker(ε^*). Thus we may define

$$\xi\colon\mathrm{Ker}((\varepsilon|_{A+I})^*)\longrightarrow\mathrm{Ker}(\varepsilon^*)$$

by $\xi(\mu)$=b. Clearly ξ is injective as f is injective. To show that ξ is surjective, let b$\in$Ker(ε^*). Therefore (0,b)$\in$Ker(g)=Im(f), so there is an x$\in$Pic(A+I) such that f(x)=(0,b). Therefore x$\in$Ker(($\varepsilon|_{A+I})^*$), and we have established the theorem.

COROLLARY 3.5. Let A$\subseteq$B be subrings of an integral domain. Then we have

$$\mathrm{Pic(A+xB[x])}\cong\mathrm{Pic(A)}\oplus\mathrm{NPicB}_1.$$

More generally,

$$\mathrm{Pic(A}[x_1,...,x_n]+x_{n+1}\mathrm{B}[x_1,...,x_n,x_{n+1}])\cong\mathrm{Pic(A}[x_1,...,x_n])\oplus\mathrm{NPicB}_{n+1}.$$

COROLLARY 3.6. If Pic(A)$\cong$Pic(B), then Pic(A+I)$\cong$Pic(B+I).

The proofs of the previous two corollaries are immediate.

REMARK 3.7. Classically in the case of rings of algebraic integers, the Picard group is considered a measure of how far a ring misses being a unique factorization domain (UFD); and, in fact, it is well-known that Pic(R) is trivial if R is a UFD. The above corollary allows us to construct many examples of rings that have trivial Picard group but are not UFD's. For example, consider the ring $\mathbf{Q}$+x$\mathbf{R}$[x]. By the above, this ring has trivial Picard group, but it is easily seen that this ring is not a UFD (consider possible factorizations of x^2); for a Noetherian example replace **R** with a finite extension of **Q.** Divisibility properties of integral domains of the form A+xB[x] have been studied extensively in [1].

We know that $\mathrm{Pic}(\mathrm{R}[x_1,...x_n])\cong\mathrm{Pic}(\mathrm{R})\oplus\mathrm{NPicR}_1\oplus...\oplus\mathrm{NPicR}_n$. We can now give a theorem which will help to elucidate the structure of these "NPics".

THEOREM 3.8. Let R be an integral domain. Then

$$\mathrm{NPicR}_n \cong\begin{cases}\mathrm{Pic}(\mathbf{Z}+\mathrm{x}_n\mathrm{R}[x_1,...,x_n]) & \text{if char(R)=0}\\ \mathrm{Pic}(\mathbf{Z}/\mathrm{p}\mathbf{Z}+\mathrm{x}_n\mathrm{R}[x_1,...,x_n]) & \text{if char(R)=p:prime.}\end{cases}$$

Proof: If char(R)=0, then $\mathbf{Z}\subseteq\mathrm{R}$, so we can apply Corollary 3.5 to obtain:

$$\mathrm{Pic}(\mathbf{Z}[x_1,...,x_{n-1}]+x_n\mathrm{R}[x_1,...,x_n])\cong\mathrm{Pic}(\mathbf{Z}[x_1,...,x_n])\oplus\mathrm{NPicR}_n.$$

But as $\mathrm{Pic}(\mathbf{Z}[x_1,...,x_{n-1}])\cong\mathrm{Pic}(\mathbf{Z})=0$ (**Z** is, of course, seminormal), Corollary 3.6 implies that

$$\mathrm{Pic}\ (\mathbf{Z}+x_n\mathrm{R}[x_1,...,x_n])\cong\mathrm{Pic}(\mathbf{Z}[x_1,...,x_n]+x_n\mathrm{R}[x_1,...,x_n])\cong\mathrm{NPicR}_n.$$

This establishes the characteristic 0 case; the proof of the characteristic p case is similar.

4. THE TWO VARIABLE CASE

Consider the polynomial ring in two variables R[x,y]. In this section, we will always assume that char(R)=0, but all the theorems will hold for prime characteristic also. In section 3, we found that

$$\mathrm{NPicR}_1\cong\mathrm{Pic}(\mathbf{Z}+\mathrm{xR[x]}).$$

Now consider the following sequence of ring homomorphisms:

$$\mathbf{Z}+\mathrm{xR[x]}\overset{\iota}{\hookrightarrow}\mathbf{Z}+\mathrm{xR[x,y]}\xrightarrow{\varepsilon_{y=0}}\mathbf{Z}+\mathrm{xR[x]}.$$

Clearly $\varepsilon_{y=0}\iota=1_{Z+xR[x]}$; therefore we can apply the Pic functor to obtain via Theorem 3.8 that

$$\mathrm{NPicR}_2\cong\mathrm{Pic}(\mathbf{Z}+\mathrm{xR[x,y]})\cong\mathrm{NPicR}_1\oplus\mathrm{Ker}((\varepsilon_{y=0})^*).$$

Consider the following Cartesian square:

$$\begin{array}{ccc}\mathbf{Z}+\mathrm{xyR[x,y]} & \xrightarrow{\iota} & \mathbf{Z}+\mathrm{xR[x,y]}\\ \downarrow & & \downarrow\\ \mathbf{Z} & \hookrightarrow & \mathbf{Z}+\mathrm{xR[x]}.\end{array}$$

Applying Theorems 3.4 and 3.8, we have the following split exact sequence:

$$0\longrightarrow\mathrm{Pic}(\mathbf{Z}+\mathrm{xyR[x,y]})\longrightarrow\mathrm{Pic}(\mathbf{Z})\oplus\mathrm{NPicR}_2\xrightarrow{\varepsilon^*_{y=0}}\mathrm{NPicR}_1\longrightarrow0.$$

As $\mathrm{Pic}(\mathbf{Z})=0$, we have that

$$\mathrm{NPicR}_2\cong\mathrm{NPicR}_1\oplus\mathrm{Pic}(\mathbf{Z}+\mathrm{xyR[x,y]}).$$

We note here that this iterative process can in general be repeated once more. Consider:

$$\mathbf{Z}+\mathrm{xR[x]}\xrightarrow{\phi:x\mapsto xy}\mathbf{Z}+\mathrm{xyR[x,y]}\xrightarrow{\varepsilon_{y=1}}\mathbf{Z}+\mathrm{xR[x]}.$$

As before, $\varepsilon_{y=1}\phi=1_{\mathbf{Z}+\mathbf{x}\mathrm{R[x]}}$, so:

$$\mathrm{Pic}(\mathbf{Z}+\mathrm{xyR[x,y]})\cong\mathrm{NPicR}_1\oplus\mathrm{Ker}((\varepsilon_{y=1})^*).$$

Now consider the Cartesian square:

$$\begin{array}{ccc} \mathbf{Z}\text{+xy(y-1)R[x,y]} & \longrightarrow & \mathbf{Z}\text{+xyR[x,y]} \\ \Big\downarrow & & \Big\downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}\text{+xR[x].} \end{array}$$

We conclude that Ker($(\varepsilon_{y=1})^*$) $\cong$Pic($\mathbf{Z}$+xy(y-1)R[x,y]), and thus

$$\text{NPicR}_2 \cong \text{NPicR}_1 \oplus \text{NPicR}_1 \oplus \text{Pic}(\mathbf{Z}\text{+xy(y-1)R[x,y]}).$$

This fact gives the following theorem:

THEOREM 4.1. NPicR$_n$ contains 2^{n-1} copies of NPicR$_1$ as direct summands.

Proof: The above can be reworked using Theorem 3.8 to show NPicR$_n$ contains two copies of NPicR$_{n-1}$ as direct summands. The rest is a trivial induction argument.

REMARK 4.2. In general, it does not seem that we are able to continue the above process and inject an infinite number of copies of NPicR$_1$ into NPicR$_2$ as direct summands. A sufficient criterion for n steps of this iterative process to be successfully completed is that (n-1)! is a unit of R. In other words, if (n-1)! is a unit of R, then we can inject n copies of NPicR$_1$ into NPicR$_2$ as direct summands, and hence in this case NPicR$_m$ contains n^{m-1} copies of NPicR$_1$ as direct summands. If R contains **Q,** then NPicR$_2$ contains an infinite number of copies of NPicR$_1$ as direct summands, see [6]). However, this condition is not necessary in general as it is trivially unnecessary when R is seminormal.

THEOREM 4.3. The homomorphism φ:$\mathbf{Z}$+xR[x,y]$\longrightarrow$$\mathbf{Z}$+xyR[x,y] given by

$$\varphi(\text{z+xr(x,y)})\mapsto\text{z+xyr(xy,y)}$$

induces an injective homomorphism φ^*:Pic($\mathbf{Z}$+xR[x,y])$\longrightarrow$Pic($\mathbf{Z}$+xyR[x,y]).

To approach the proof of this theorem, we need to develop some facts about the structure of $\mathbf{Z}$+xyR[x,y].

LEMMA 4.4. $\mathbf{Z}$+xyR[x,y]=I^x+I^y with I^y={p $\in$ $\mathbf{Z}$ + xyR[x, y]|every monomial of p has deg$_y$ $\geq$ deg_x}$\bigcup${0} and I^x={p $\in$ $\mathbf{Z}$ + xyR[x, y]|every monomial of p has deg$_x$ $>$ deg_y}$\bigcup${0}

Proof: Trivial.

REMARK 4.5. $\mathbf{Z}$+xyR[x,y]=I$^y\oplus$I^x as an abelian group, and additionally, a,b$\in$I^x $\Rightarrow$ab$\in$I^x. The same multiplicative self-closure holds for I^y.

LEMMA 4.6. Any f$\in$$\mathbf{Z}$+xyR[x,y]=I$^y\oplus$I^x can be written as (f_1, f_2) with f_1 $\in$I^y and f_2 $\in$I^x. If $(f_1, f_2)(g_1, g_2)$=(k,0), then either f_2=0 or g_2=0.

Proof of Lemma 4.6: Assume both f_2, $g_2 \neq 0$. Pick m_f,m_g nonzero monomials of f (respectively g) such that $\deg_x - \deg_y$ is maximized (this maximum degree difference is positive as f_2, g_2 are nonzero), and we stipulate that if there is more than one monomial of f (resp. g) with this property, then we pick the one with the greatest total degree. Thus we can write f and g as f=$\widehat{f}+m_f$ and g=$\widehat{g}+m_g$. Then fg=$\widehat{f}\,\widehat{g} + m_f\,\widehat{g} + m_g\,\widehat{f} + m_g m_f$. Clearly the monomial $m_g m_f$ maximizes the difference $\deg_x$(m)$-\deg_y$(m) among all monomials of fg, and among the maximizing monomials, it has the greatest total degree. Therefore fg has at least one nonzero monomial in I^x, which gives a contradiction.

Proof of Theorem 4.3: The map φ^*:Pic($\mathbf{Z}$+xR[x,y])$\longrightarrow$Pic($\mathbf{Z}$+xyR[x,y]) is given by φ^*([I])=[I^e], where I^e denotes the extension of the ideal I with respect to the homomorphism φ. We first note that φ^* is well-defined, for if [I]=[J], then I=aJ with a in the quotient field of $\mathbf{Z}$+xR[x,y]. But I^e=(aJ)^e=(a)^{e}J^e. Therefore [I^e]=[J^e], so we have established that φ^* is well-defined. Also note that as (IJ)^e=$\mathrm{I}^e\mathrm{J}^e$, φ^* is a homomorphism from Pic($\mathbf{Z}$+xR[x,y]) to Pic($\mathbf{Z}$+xyR[x,y]). It now suffices to show that φ^* is injective. Let [I]$\in$ Pic($\mathbf{Z}$+xR[x,y]) be such that φ^*([I])=[$\mathbf{Z}$+xyR[x,y]]. Without loss of generality, we pick our representative I of [I] such that I$\subseteq\mathbf{Z}$+xR[x,y]. Therefore I^e=(p)$\subseteq\mathbf{Z}$+xyR[x,y].

Consider $(\mathrm{p}^c)^{-1}$I, where p^c is the image of p under the change of variables x$\mapsto \frac{x}{y}$ and y$\mapsto$y. Therefore as p^c is in the quotient field of $\mathbf{Z}$+xR[x,y], $(\mathrm{p}^c)^{-1}$I=J is an invertible ideal in [I]. Note that φ^*([J])=[$\mathbf{Z}$+xyR[x,y]]. To show that [I] is the principal class, it is enough to show that J=$\mathbf{Z}$+xR[x,y].

As J is invertible, it is finitely generated, so we can write J=$(f_1, f_2, ..., f_n)$. We can also find generators of J^{-1}=$(\alpha_1, \alpha_2, ..., \alpha_n)$ such that $\alpha_i f_j \in\mathbf{Z}$+xR[x,y] for all $1 \leq i, j \leq n$ and $\alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_n f_n$=1. By the above, $\varphi^*([\mathrm{J}^{-1}]) = (\varphi^*([\mathrm{J}]))^{-1}$. As φ^*([J])=[$\mathbf{Z}$+xyR[x,y]], this means $(\varphi^*([\mathrm{J}]))^{-1}$=[$\mathbf{Z}$+xyR[x,y]]. Combining the above observations we obtain two important facts. First of all, the image of $f_i, \alpha_i \in\mathbf{Z}$+xyR[x,y] for all $1 \leq i \leq n$, and secondly, the image of $\alpha_i f_j \in \mathrm{I}^y$ for all $1 \leq i, j \leq n$.

We denote the image of α_i as $(\alpha_{i,1}, \alpha_{i,2})$ and the image of f_i as $(f_{i,1}, f_{i,2})$ with the first coordinate in I^y and the second in I^x. By the above remark,

$$(\alpha_{i,1}, \alpha_{i,2})(f_{j,1}, f_{j,2}){=}(\gamma_{i,j},0) \text{ for all } 1 \leq i, j \leq n.$$

Now assume that we have $f_{i,2} \neq 0$ for some i. Therefore, as

$$(\alpha_{j,1}, \alpha_{j,2})(f_{i,1}, f_{i,2}) = (\gamma_{j,i}, 0) \text{ for all } 1 \leq j \leq n,$$

this implies (by Lemma 4.6) that $\alpha_{j,2} = 0$ for all $1 \leq j \leq n$. Also note that for all $j, (\alpha_{j,1}, 0)(f_{i,1}, f_{i,2}) = (\gamma_{j,i}, 0)$ implies that $\alpha_{j,1} f_{i,2} \in \mathrm{I}^y$ because of the additive closure of I^y. Therefore, as $\deg_x$(m)>$\deg_y$(m) for all monomials m of $f_{i,2}$, we have that $\deg_y$(n)>$\deg_x$(n) for all monomials n of $\alpha_{j,1}$. In particular, no term on the left hand side of the equation:

$$(\alpha_{1,1}, 0)(f_{1,1}, f_{1,2}) + ... + (\alpha_{i,1}, 0)(f_{i,1}, f_{i,2}) + ... + (\alpha_{n,1}, 0)(f_{n,1}, f_{n,2}) = 1$$

has a constant term, which is an obvious contradiction. Thus $f_{i,2} = 0$ for all i, and the same proof gives that $\alpha_{i,2} = 0$ for all i. Therefore, as the images of both α_i and f_i are in I^y, the change of variables x$\mapsto \frac{x}{y}$ shows that α_i and f_i are in $\mathbf{Z}$+xR[x,y] for all i. Hence $(p^c)^{-1}I=\mathbf{Z}+xR[x,y]$ and the theorem is established.

LEMMA 4.7. If G,H,K are groups such that $G\cong H\oplus K$ and there exists an injective homomorphism $\psi:G\longrightarrow H$, then G contains (an isomorphic copy of) $\bigoplus_{i=1}^{\infty}K$.

Proof: $G=K_1\oplus H_1$ with K_1 and H_1 isomorphic to K and H respectively. Then H_1 has a subgroup $G_1=K_2\oplus H_2$ with K_2 and H_2 isomorphic to K and H respectively. Thus $K_1\oplus K_2\oplus H_2$ is a subgroup of G. Continuing this process we get that $K_1\oplus K_2\oplus K_3\oplus...$ is a subgroup of G isomorphic to $\bigoplus_{i=1}^{\infty}K$.

Combining our previous results with the above lemma, we obtain the following theorem.

THEOREM 4.8. $NPicR_2 \supseteq Pic(\mathbf{Z}+xyR[x,y])\oplus(\bigoplus_{i=1}^{\infty}NPicR_1)$. In general, $NPicR_n \supseteq \bigoplus_{i=1}^{\infty}NPicR_{n-1} \supseteq \bigoplus_{i=1}^{\infty}NPicR_1$.

REMARK 4.9. From a group-theoretic standpoint, there is no reason to suspect that $NPicR_2 \cong Pic(\mathbf{Z}+xyR[x,y])\oplus(\bigoplus_{i=1}^{\infty}NPicR_1)$. For example, consider the abelian groups

$$G=\mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}\oplus(\textstyle\bigoplus_{i=1}^{\infty}\mathbf{Z}/4\mathbf{Z}),$$
$$H=\mathbf{Z}/2\mathbf{Z}\oplus(\textstyle\bigoplus_{i=1}^{\infty}\mathbf{Z}/4\mathbf{Z}), \text{ and}$$
$$K=\mathbf{Z}/2\mathbf{Z}.$$

In this example it is certainly true that $G\cong H\oplus K$, but G merely contains the direct sum of an infinite number of copies of K as a subgroup. For a torsion-free example, replace each $\mathbf{Z}/2\mathbf{Z}$ above with $\mathbf{Z}$ and replace each $\mathbf{Z}/4\mathbf{Z}$ with $\mathbf{Z}_3$ (the 3-adic integers). However, we have deduced that $NPicR_n$ is either 0 (the seminormal case) or very large.

5. THE RELATIONSHIP BETWEEN Pic(R) AND $NPicR_1$

As we have seen in the previous sections, $NPicR_1 \cong Pic(\mathbf{Z}+xR[x])$ when char(R)=0. In this context, it would seem natural to induce a map from $NPicR_1$ to Pic(R) from the obvious evaluation map from $\mathbf{Z}+xR[x]$ to R. Any evaluation map induces a well-defined map from $NPicR_1$ to Pic(R). Unfortunately, this homomorphism is neither injective nor surjective in general.

THEOREM 5.1. If R is not seminormal, then $NPicR_1$ is not finitely generated.

In order to prove Theorem 5.1 we will need the following lemma.

LEMMA 5.2 For each $n\geq1$, let $I_n=(1+sx^n,s^2x^{2n})$ with s a fixed nonseminormal element of K. Then each I_n represents a distinct equivalence class in Pic(R[x]).

Proof of Lemma 5.2: Recall that $I_n^{-1}=(1-sx^n,s^2x^{2n})$. So if $[I_n]=[I_m]$ for some n$\neq$m, then $[I_n I_m^{-1}]$ is the principal class. A system of generators for $J=I_nI_m^{-1}$ is $\{1+sx^n-sx^m-s^2x^{n+m}, s^2x^{2m}+s^3x^{2m+n}, s^2x^{2n}-s^3x^{2n+m}, s^4x^{2n+2m}\}$. Immediately we see that if J is principally generated, then it must be generated by a monomial in x (as J contains a monomial). Also, as J contains an element with a constant coefficient, the generating monomial r must have degree 0 (i.e., it must be in K). Since r is generated by the above set over R[x], it must be in R. But this is a contradiction as the first element of the generating set is not in R[x]. Thus we have established the lemma.

Proof of Theorem 5.1: Let $\{I_n\}$ be as in the previous lemma. We have shown that each I_n represents a distinct equivalence class in Pic(R). As NPicR_1 is an abelian group, it suffices to show that the group generated by $\{I_n\}$ is not finitely generated.

Case 1: s is such that ms$\notin$R for all nonzero m$\in\mathbf{Z}$.

Assume that $I_{n_1}^{a_1}I_{n_2}^{a_2}...I_{n_r}^{a_r}=(1)$. This is impossible if any of the a_is are nonzero, as the left hand side of this equation would then contain elements with nonseminormal linear terms (by the binomial theorem), and therefore the above product cannot be principally generated by the same argument as in the proof of Lemma 5.2. Thus in this case, each I_n has infinite order and $\{I_n\}$ forms a basis for this subgroup which is infinitely generated free abelian. Hence in case 1, NPicR_1 is not finitely generated.

Case 2: There is a nonzero m$\in\mathbf{Z}$ such that ms$\in$R.

In this case, $I_n^m=(1)$ for all n. To see this, we first note that by the binomial theorem, $I_n^m \subseteq$R for any n. We also note that $(I_n^m)^{-1}=(I_n^{-1})^m$ is also contained in R by the binomial theorem. Therefore I_n^m is an ideal in R whose inverse is in R, and thus $I_n^m=(1)$ for all n$\geq$1. Therefore as I_n has finite order for all n, the group generated by $\{I_n\}$ cannot be finitely generated. Hence we have established Case 2 and the theorem.

EXAMPLE 5.3. Let $R=\mathbf{Z}[\sqrt{-5}]$, the ring of integers of the field $\mathbf{Q}(\sqrt{-5})$, and $T=R[s^2,s^3]$. It is well-known that the class number of R is 2 and that $I=(2,1+\sqrt{-5})$ is a representative of the non-trivial coset in Pic(R). I is an invertible ideal of R; hence it extends to an invertible ideal of T (which we shall denote by I^*). Here we will show that I^* is not in the image of $\phi^*:\mathrm{NPicT}_1 \longrightarrow \mathrm{Pic(T)}$, where $\phi:\mathbf{Z}+xT[x]\longrightarrow T$ by $\phi(f(x))=f(a)$ with a an element of R.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}+xT[x] & \xrightarrow{\varepsilon_{s=0}} & \mathbf{Z}+xR[x] \\ \phi\downarrow & & \downarrow\phi_R \\ T & \xrightarrow{\varepsilon_{s=0}} & R. \end{array}$$

In this diagram, the two homomorphisms denoted $\varepsilon_{s=0}$ set the nonseminormal element equal to 0; ϕ is just evaluation of x at some element of R; and ϕ_R is just this evaluation map restricted to $\mathbf{Z}$+xR[x]. It is easy to see that this is indeed a commutative diagram. This gives rise to the following commutative diagram:

$$\begin{tikzcd} \mathrm{NPicT}_1 \arrow[r] \arrow[d] & \mathrm{NPicR}_1 \arrow[d] \\ \mathrm{Pic(T)} \arrow[r] & \mathrm{Pic(R)}. \end{tikzcd}$$

Now assume J is in NPicT_1 such that [J] maps to [I*] in Pic(T). Clearly [I*] maps to [I] in Pic(R) (as the image of I* must contain I but be non-trivial, and I is maximal). Notice however that if we "follow" [J] the other way, we get [R] as NPicR_1 is trivial (as R is integrally closed, hence seminormal). So we have a contradiction. Hence we have that these evaluation-induced maps are not generally surjective.

It is easier to see that these maps are not generally injective. Modifying Example 2.3, we can let $I=(1\text{-}sx(x\text{-}a),s^2(x(x\text{-}a))^2)$ for an example of a non-trivial invertible ideal that is in the kernel of the map induced by evaluation at a.

ACKNOWLEDGEMENTS

I would like to express my most sincere gratitude to Dr. David F. Anderson for starting me on this project and helping to mold my career. Also I would like to thank Dr. David E. Dobbs for his continued support and advice.

REFERENCES

1. D. D. ANDERSON, D.F ANDERSON, and M. ZAFRULLAH, Rings between D[x] and K[x], *Houston J. Math.* **17**(1991), 109-129.
2. D. F. ANDERSON, The Picard group of a monoid domain, *J. Algebra* **115** (1988), 342-351.
3. H. BASS, "Algebraic K-Theory," Benjamin, New York, 1968.
4. H. BASS and M. P. MURTHY, Groethendieck groups and Picard groups of group rings, *Ann. of Math.* **86**(1967), 16-73.
5. J. W. BREWER and D. L. COSTA, Seminormality and projective modules over polynomial rings, *J. Algebra* **58**(1979), 208-216.
6. B. H. DAYTON and C. A. WEIBEL, Module structures on the Hochschild and cyclic homology of graded rings, *NATO ASI Series C* **407**(1993).
7. R. GILMER and R. HEITMANN, On Pic(R[x]) for R seminormal, *J. Pure Appl. Algebra* **16**(1980), 251-257.
8. I. KAPLANSKY, "Infinite Abelian Groups," Univ. of Mich. Press, Ann Arbor, 1969.
9. J. MILNOR, "Introduction to Algebraic K-Theory," Ann. of Math. Studies 72, Princeton Univ. Press, Princeton, N.J., 1971.
10. R. G. SWAN, On seminormality, *J. Algebra* **67**(1980), 210-229.