Research Statement

Kevin Dilks

October 24, 2017

1 Introduction

My mathematical research area is in combinatorics, and more specifically, algebraic combinatorics.

Algebraic combinatorics is mainly focused on the interplay between combinatorics and topics from abstract algebra like representation theory. This relationship works in both directions. Sometimes questions from abstract algebra can be reduced to a combinatorial problem, like how the representation theory of the symmetric group is encoded by the combinatorics of discrete objects called *tableaux*. Other times, questions arise by taking a combinatorial object and trying to imbue it with additional algebraic structure. I am also particularly interested in topics like Coxeter groups, which have a rich geometric structure, and where there is a threeway relationship between algebra, geometry, and combinatorics.

In the rest of this research statement, I will summarize the projects I have been working on most recently, the projects I am working towards, and my long-term research goals. The topics that I have been working on can be broadly grouped into the three categories. One is dynamical algebraic combinatorics, which studies actions on combinatorial objects from algebraic combinatorics. The second is Baxter permutations, a class of permutations given by a pattern avoidance condition that is in bijection with a number of other combinatorial objects. Lastly, there is q-gamma nonnegativity, which is trying to find and explain multivariate generalizations of certain expansions of a polynomial that arises naturally when studying flag simplicial complexes.

2 Dynamical Algebraic Combinatorics

In combinatorics, one of the most basic problems is to give a simple formula that enumerates how many objects are in a certain set. The number of subsets of the numbers $\{1, \ldots, n\}$ is 2^n, the number of those subsets of size k is $\binom{n}{k}$, and the number of triangulations of a regular $(n+2)$-gon is given by the nth Catalan number. Once an enumeration is known, and you know that two sets have the same enumeration, one might endeavor to find an explicit bijection mapping one set to the other. There are over 200 different combinatorial families who enumeration is given by the Catalan numbers [21], and many of them have vastly different structures.

In dynamical algebraic combinatorics, as featured in the June/July 2017 AMS Notices [22], the goal is to better understand how combinatorial actions and natural symmetries are preserved under these explicit bijections. For example, some combinatorial families enumerated by the Catalan number have an obvious rotation action (like triangulations of a regular $(n+2)$-gon), and for others it is less clear what this action should correspond to (binary trees on $n+2$ nodes do not obviously rotate).

Most recently, my work has been focused on establishing a connection between increasing tableaux with the associated action of K-promotion, and order ideals in the Cartesian product of three chains with the associated action of rowmotion. I have also been working on extending these results to make a connection between increasing labelings of an arbitrary poset with a generalization of K-promotion and order ideals in an associated poset with rowmotion [10].

A *Young diagram* is an arrangement of boxes corresponding to a *partition* (a weakly increasing sequence of non-negative integers, usually with a specified sum). Standard representation theory tells us that the number of conjugacy class in S_n (and thus the number of irreducible representations) is equal to the number of partitions (conjugacy classes in S_n are given by their cycle structure). The dimension of the representation for a given partition is the number of ways we can bijectively label the boxes in the corresponding Young diagram 1 through n so the labels are strictly increasing along rows and columns, and these labelings are called *Young tableaux.* These can also be thought of as linear extensions of some underlying partial order structure. There is a natural action on Young tableaux (and more generally, linear extensions of a poset) called *promotion*, which can be described either in terms of sliding boxes, or as a series of local involutions [20].

In the equivariant K-theory of the Grassmannian, the relevant objects end up being *increasing tableaux.* These are also labelings of a Young diagram with integers that must strictly increase along rows and columns, but we are allowed to use a number more (or less) than once. An action analagous to promotion was defined (called *K-promotion*) in terms of box-sliding operations [24][17], and was later shown to also be describable as a series of local involutions [8]. It was observed that for increasing tableaux of square shape, the sizes of the orbits were typically equal to (or divisors of) an expected order, but occasionally orbit sizes would be small multiples of this expected order.

A partially ordered set (or *poset*) is a set P and a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. Given a poset P, one can consider *order ideals*, which are subsets $I \subseteq P$ that are 'closed under going down' (i.e., if $y \in I$ and $x \leq y$, then $x \in I$). The order ideals of a poset form a distributive lattice with the operations of union and intersection, and is often denoted $J(P)$. Many times, objects of interest can be thought of as order ideals in a poset. For example, Young diagrams can be thought of as order ideals in the Cartesian product $\mathbb{N} \times \mathbb{N}$.

One action that has been frequently studied on order ideals of a poset is *rowmotion.* Rowmotion has a global description, where we take the minimal elements not in our order ideal, and let them generate a new order ideal. They also have a description as a series of local involutions called *toggles*, which try to add/remove an element from an order ideal if the result will still be an order ideal. Rowmotion is the result of toggling every element exactly once, going from top to bottom. When looking at rowmotion on a product of three chains, it was observed that the sizes of the orbits were typically equal to (or divisors of) an expected order, but occasionally orbit sizes would be small multiples of this expected order.

In my paper with Oliver Pechenik and Jessica Striker [8], we managed to connect these two phenomenon. We established a bijection between increasing tableaux of square shape and order ideals in a product of three chains, which while relatively simple, seems to have been under-utilized. But more importantly, we were able to show this bijection carried K-promotion on increasing tableaux to an action we called *hyperplane promotion* on order ideals in the product of three chains. This action corresponds to sweeping through the product of three chains at an angle and toggling every element once. While not exactly equal to rowmotion, using a generalization of the method of Striker and Williams [23], we were able to show that rowmotion and hyperplane promotion were conjugate actions, and thus have the same orbit structure. We were also able to derive a number of nice results on increasing tableaux by exploiting the connection to order ideals in a product of three chains and it's 3-fold symmetry.

In my most recent work with Jessica Striker and Corey Vorland, we deconstructed the proof of the above work to figure out which properties on each side of the bijection were necessary, and which things could be relaxed and extended to a more general setting. We were able to show that given a poset P and a function giving a range of possible values for each $p \in P$ satisfying a local condition, one could construct an associated poset $\Gamma_1(P,R)$ whose order ideals were in bijection with increasing labelings of the original poset. We were also able to define a generalization of K-promotion on these more arbitrary increasing labelings, and show that it exactly corresponded with toggling in a particular order on $\Gamma_1(P,R)$. In the case where the restrictions are those induced by saying you only want to be able to use labels from 1 up to some value q, we were able to show that this toggling action on $\Gamma_1(P,R)$ would be conjugate to rowmotion.

3 Baxter Permutations

Baxter permutations are a class of permutations that can be given by a pattern avoidance condition. They first arose in the work of Glen Baxter in the 1960s, who was looking at permutations of fixed points in compositions of continuous commuting functions [1]. They were not studied again until 1978, when Chung, Graham, Hoggatt, and Kleiman [2] showed that the enumeration of the number of Baxter permutations of length n was given by

$$B(n) = \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}}.$$

Shortly thereafter, Mallows [16] showed that the k^{th} summand in this expression gave the number of Baxter permutations with k descents (if we write a permutation in one-line notation as $w = w_1 \dots w_n$, a *descent* is an instance where $w_{i+1} < w_i$). However, these proofs relied on algebraic manipulations of a multi-term recurrence relation, and weren't enlightening. In 1981, Viennot was able to give a bijective proof of this enumeration formula [25]. He showed that Baxter permutations of length n with k descents were in bijection with non-intersecting triples of lattice path with specified endpoints, making the enumeration formula a consequence of his work on non-intersecting lattice paths with Gessel [14].

Later work was done by Cori, Dulucq, and Viennot to establish bijections between Baxter permutations with pairs of binary trees satisfying a compatibility relationship, and also with certain shuffles of parenthesis systems [3]. In the late 1990s, Dulucq and Guibert extended those bijections to a class of stackwords (in bijection with standard Young tableaux of shape $3 \times n$) [11]. They also strengthened the bijection between Baxter permutations and non-intersecting triples of lattice paths to get more refined enumeration formulas for the number of Baxter permutations with respect to certain statistics.

More recently, Reading and Law showed that Baxter permutations were in bijection with diagonal rectangulations and a similar class of permutations called twisted Baxter permutations [15]. Using the machinery of Reading's work on lattice congruences of the weak order [18], they were able to put a Hopf algebra structure on twisted Baxter permutations and diagonal rectangulations. Felsner, Fusy, Orden, and Noy also provided additional bijections to objects like planar maps and quadrangulations [12].

One thing that I have worked with are Baxter permutations (and other combinatorial objects in bijection with them) that were fixed under a natural involution [4]. If one can show that these bijections commute with the natural involutions on the other objects, then the enumeration of objects fixed under involution should be the same for all of them. I was able to show that the bijections between all other known Baxter objects commuted with their natural involutions. Additionally, I was able to give a simple enumerative formula, and an interesting new instance of Stembridge's $q = -1$ phenomenon. In particular, I showed that for

$$\Theta_{k,n-k}(q) := \frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q},$$

that $\Theta_{k,n-k}(q)$ is a polynomial in q with positive integer coefficients such that $\Theta_{k,n-k}(1)$ gives the number of Baxter permutations of length n with k descents, and $\Theta_{k,n-k}(-1)$ gives the number of Baxter permutations of length n with k descents fixed under conjugation by the longest element (where $\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$, $[m]!_q = [m]_q[m-1]_q \dots [1]_q$, and $[j]_q = 1 + q + \dots + q^{j-1}$).

Additionally, using the method of generating trees, I was able to give an enumerative formula for the number of Baxter permutations fixed under a 90° of the permutation matrix [6]. While all Baxter objects have a natural order 2 symmetry, this finer order 4 symmetry does not seem to correspond to anything natural other than Baxter permutations. I showed that Baxter permutations of length n can only be fixed under this quarter turn rotation when $n = 4k + 1$, and then the number of them will be $2^k C_k$, where C_k is the kth Catalan number.

4 q-Gamma Nonnegativity

Given a simplicial polytope, there is a natural encoding of the number of faces of each dimension called the h polynomial. The face numbers for a simplicial polytope satisfy the Dehn-Sommerville equations, which when translated to the h-polynomial just means that it has symmetric coefficient sequence (i.e., if $h(t) = \sum_{i=0}^{n} a_i t^i$, then $a_i = a_{n-1}$). In 1980, Stanley showed that the h-polynomial of a convex simplicial polytope has unimodal coefficients (i.e., $h_i \leq h_{i+1}$ for $i < \lfloor n/2 \rfloor$).

More recently, Gal has conjectured something stronger. Any polynomial $h(t)$ with symmetric coefficient sequence can be expanded uniquely as $h(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k t^k (1+t)^{n-2k}$, and Gal's conjecture says that this expansion of the h-polynomial for any flag generalized homology sphere (which includes simplicial polytopes) must have nonnegative coefficients γ_k [13]. Many expansions of this type are known, including some I was able to derive (with Kyle Petersen and John Stembridge) for an affine generalization of the Coxeter complex [9].

One topic I have been exploring is finding q-analogs of expansions related to Gal's conjecture [5]. Some overlapping results were concurrently discovered independently by Michelle Wachs and Christan Krattenthaler, and a joint publication is in preparation [7]. Oftentimes, the h-polynomial for simplicial polytopes of combinatorial significance can be given as a generating function over a related set of combinatorial objects with respect to some statistic like number of descents.

For example, the h-polynomial corresponding to the associahedron is given by

$$\mathrm{Cat}_n(t) = \sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n}{k+1}}{\binom{n}{1}} t^k,$$

and can be expanded as

$$\mathrm{Cat}_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_k \binom{n-1}{2k} t^k (1+t)^{n-1-2i}.$$

In my thesis, I showed that the generating function for the q-Narayana numbers,

$$C(n,q,t) = \sum_{k=0}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} q^{k^2+k} t^k,$$

has a similar kind of multivariate expansion,

$$\mathrm{Cat}_n(q,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_k(q^2) \begin{bmatrix} n-1 \\ 2k \end{bmatrix}_q q^{k(k+2)} t^k \prod_{i=k}^{n-1-k} (1+tq^{2k+2}).$$

5 Short Term Projects

One project that I am interested in working on is finding applications of the connection between increasing labelings of a poset and order ideals of an associated poset. My work with Pechenik and Striker involved a connection to increasing tableaux, motivated by a connection to equivariant K-theory of the Grassmannian. However, we only considered the case of square Young diagrams, and we should now be able to consider the case of arbitrary shapes. My work with Striker and Vorland [10] is also typically framed as being given a poset and restrictions for entries in an increasing labeling, and then the associated poset is constructed from that information. However, in some cases it may be possible to do the reverse, and realize that an object with a description in terms of order ideals of one poset may be more naturally understood in terms of increasing labelings of a different poset. There are many combinatorial objects (like Baxter permutations) that are in bijection with plane partitions or order ideals in some other poset, and this may yield a new way to realize these objects and discover more structure behind them.

Additionally, I'm interested in expanding upon the underlying group theory that my work in dynamical algebraic combinatorics relies on. Showing rowmotion and promotion are conjugate relies on dividing a poset into columns C_1 through C_n where columns only have covering relations to adjacent columns, and then using the fact that all Coxeter elements are conjugate when the Coxeter diagram is acyclic. I would be interested in exploring if there are situations where we can have a poset embedded on a cylinder or a sphere where the columns C_1 through C_n only have covering relations to adjacent columns modulo n. In this case, the corresponding Coxeter diagram would not be acyclic, but it is still possible that the relevent Coxeter elements we need still lie in the same conjugacy class. Additionally, there is an action associated to promotion (as originally defined on linear extensions of a poset) called evacuation, which is always an involution. The definition of evacuation generalizes to promotion and rowmotion of order ideals on a poset, so I am interested in further exploring what this involution corresponds to.

6 Long Term Projects

One general idea I would like to understand better is what has been called the "resonance phenomenon". There are a number of situations in combinatorics where a naturally defined cyclic action may typically have a large and hard to compute order, but will have a small and predictable order for well-behaved families. The resonance phenomenon occurs when one generalizes slightly beyond the well-behaved families, and the order is a small multiple of what the more well-behaved examples would have suggested it should be. In some sense, this suggests that the cyclic action has a deeper structure, and that we only understand it in special cases when part of that structure trivializes.

I am also interested in continuing to explore the cyclic sieving phenomenon [19]. Shortly put, an action on a a set exhibits the cyclic sieving phenomenon if the orbit structure of the action can be determined by evaluating a natural generating function associated to the set at certain roots of unity corresponding to the order of the action. One goal would be to find more examples of the cyclic sieving phenomenon in combinatorial families like Baxter permutations, where cyclic symmetry may not be immediately apparent. Additionally, I would be interested in figuring out how to extend known instances of the cyclic sieving phenomenon (which typically occur in cases where the order of the action is small and predictable) to cases where the resonance phenomenon occurs.

Many problems in combinatorics, especially the ones that I enjoy studying, are amenable to computer exploration. I not only use Sage for my own research, but I am also an active developer in the Sage community. I plan to continue to try and integrate the objects and bijections that I am working on into the Sage source code, so that other researchers may more easily build upon my work.

Lastly, I am generally interested in trying to find multivariate generalizations for combinatorial formulas, and use that as a guide to try and find additional structure in combinatorial objects. For example, I feel that there is still much to be done with respect to q-gamma nonnegativity. I currently have results and conjectures for specific examples. Some of these examples, like the associahedron, have a geometric connection as a q-analog for the h-polynomial. In these cases, I would like to find some larger family of objects, like Coxeter complexes or flag nestohedra, where this additional q parameter can be uniformly understood. In other examples, like the ones related to Baxter permutations and involutions, there is currently no known geometric object where these arise as the h-vector, but the results on gamma nonnegativity suggest that such a geometric structure should exist.

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