Total Promotion

Kevin Dilks

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Abstract

Promotion is a bijection from the set of linear extensions of a finite partially ordered set to itself. Total promotion is when promotion is iterated p times, where p is the number of elements in the poset. Certain special classes of posets, such as rectangles, have all their linear extensions fixed by total promotion. In this paper, we explore total promotion for general posets.

1 Introduction

Promotion was first studied along with a related operator called evacuation by Schützenberger [6]. Though originally intended primarily for Young tableaux, the definitions are general enough to be easily extended to general posets, and were presented by Schutzenberger in this level of generality. For Young diagrams, promotion is an operation that takes one standard Young tableaux to another via jeu de taquin slides. Total promotion is when the operation of promotion is done n times, where n is the number of boxes in the Young diagram. It is known that any Young tableuax associated to a rectangle is fixed by total promotion. This result is credited to Schützenberger as it follows immediately from his work, though the result was never actually stated. Edelman and Greene [1] showed that for staircases, total promotion gives the transpose of the original labeling. Later, Haiman [2] showed that, along with rectangles, all labelings of shifted double staircases and shifted trapezoids are also fixed by total promotion. Haiman and Kim [3] almost concurrently showed that among connected shapes, staircases were the only symmetric shapes that were transposed by total promotion, and rectangles, shifted staircases, and shifted trapezoids were the only shapes fixed by total promotion. From these results, it is natural to ask if there are any other infinite classes of posets like rectanges, shifted double staircases, and shifted trapezoids that are fixed by total promotion, and also what can be said about which posets in general are fixed by total promotion.

2 Basic Definitions

We will lay out the basic definitions and groundwork for studying promotion, mostly following the same path and notation as Stanley [8]. Let $(P, <)$ be a partially ordered set with p elements. We say that b covers a, or $a \lessdot b$ if $a < b$ and there is no c such that $a < c < b$. The order dual of P (denoted $(P^*, >)$) is the poset with the same vertex set as P, but we have $x \leq y$ in P^* if and only if $y \leq x$ in P. We say that P is self-dual if it is isomorphic to its order dual. A linear extension (or labelling) of a p-element poset P is a bijective mapping $f : P \to \{1, \dots, p\}$ such that $x \leq y$ in P implies that $f(x) \leq f(y)$ in $\mathbb{Z}$. Let $\mathcal{L}(P)$ be the set of linear extensions of P. The operation of promotion, which takes one linear extension to another, is best defined algorithmically.

Definition 1 (Promotion)**.** Promotion is given by the following algorithm:

1. Find the element whose label is 1, call it x_1. Remove its label.
2. If x_i is unlabeled and not a maximal element, then let x_{i+1} be the element that covers x_i with the smallest label. Then we "promote" the label of x_{i+1} to x_i (also known as a *jeu de taquin slide*), so now x_{i+1} is left unlabeled.
3. Repeat this process until a maximal element is left unlabeled, call it x_n. Then give x_n the label $p+1$, and reduce the label of every element in the poset by 1 (so the labels are restandardized to $1, \dots, p$).

If $L \in \mathcal{L}(P)$ is the initial linear extension, then we define $\partial(L)$ to be the new linear extension obtained by promotion. It is clear that $x_1 \lessdot \dots \lessdot x_n$ will be a maximal chain, and we call this the *chain of promotion*. An example of promotion is shown in Figure 1 with the chain of promotion highlighted.

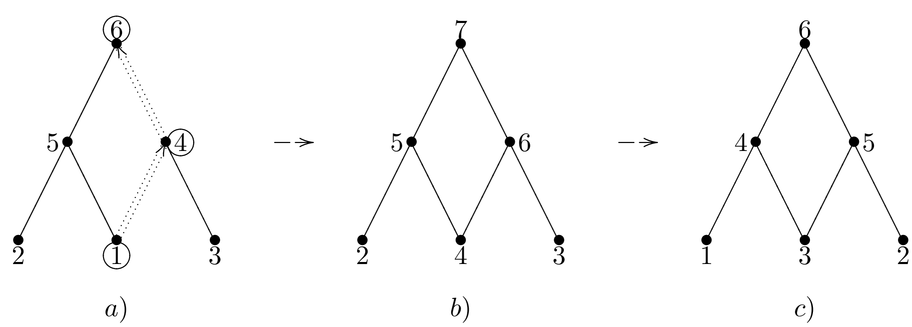


Figure 1: a) The chain of promotion b) The result of sliding c) The relabeling

Evacuation is another operator, which is best described algorithmically in terms of promotion.

Definition 2 (Evacuation)**.** Evacuation is given by the following algorithm:

1. Apply promotion to P. If x is the maximal element in the chain of promotion, then we "freeze" x with the label p, and let $P_1 := P - x$.

2. Now, apply promotion the $p-1$ element poset P_1. If y is the maximal element in the chain of promotion, then we freeze y with the label $p-1$ and let $P_2 := P_1 - y$.

3. Repeat this process until all the labels of P have been frozen, and you are left with a new linear extension of P.

If $L \in \mathcal{L}(P)$ is the initial linear extension, then we will define $\epsilon(L)$ to be its evacuation. In Figure 2 is an example of evacuation, with the chain of promotion highlighted and the "frozen" elements circled.

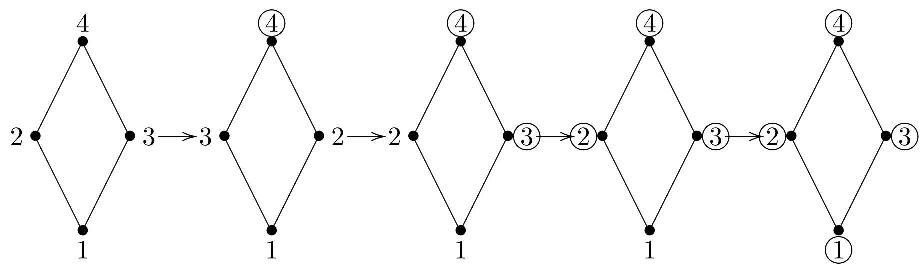


Figure 2: An example of evacuation.

Both of these operators have a very similar dual operator. For dual promotion, we remove the the largest label, and repeatedly slide the largest label of all the elements that the unlabeled element covers. When we reach a minimal vertex, we give it the label 0 and then add 1 to all the labels to get a standardized linear extension. It is not hard to see that dual promotion is in fact the inverse to promotion, and thus we shall denote the dual promotion of a linear extension $L \in \mathcal{L}(P)$ as $\partial^{-1}(L)$. Dual evacuation is defined the same as evacuation, but with repeated application of dual promotion. We shall denote dual evacuation as ϵ^*.

Schutzenberger proved some basic relations between promotion and evacuation, which are given here.

Theorem 1. *[6] For P a p-element poset,*

1. $\epsilon^2 = 1$, *i.e., evacuation is an involution,*
2. $\epsilon^* \epsilon = \partial^p$,
3. $\partial \circ \epsilon = \epsilon \circ \partial^{-1}$.

We will present proofs of these three facts that are rather intuitive once the necessary machinery has been built. We will do so using growth diagrams, following the example of Fomin in Appendix 1 of [7].

First, we consider an alternate intepretation of promotion. An ideal $I \subseteq P$ is a subset of P such that if $x < y$ and $y \in I$, then $x \in I$. The ideals of P can be given a partial order by containment, and we call this the *lattice of order ideals*, denoted by $J(P)$. We can establish a bijection between linear extensions of P and maximal chains in the lattice of order ideals. If we're given a linear extension of P, we can define x_i to be the ideal of all elements of P with labels less than or equal to i. Then $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_p$ is a maximal chain in $J(P)$. Conversely, given a maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_p$ in $J(P)$, we can define a linear extension by giving the element that's in x_i but not x_{i-1} the label i. It is easy to see that these two correspondences are inverse to each other, and thus define a bijection. So now we may define promotion as a permutation of the set of maximal chains in $J(P)$.

Growth diagrams are best defined with a picture. A growth diagram is an array on maximal chains that is very useful in understanding promotion and evacuation. We start with a maximal chain, which is written on a diagonal up and to the right. Then to the right of any chain, we write the maximal chain that corresponds to the promotion of the previous one. We also add a line connecting the i^{th} element of one maximal chain to the $(i-1)st$ element of the next maximal chain. We can also extend the diagram in the opposite direction by placing the dual promotion of a maximal chain to it's left, and adding the associated lines. As promotion has finite order, the pattern of maximal chains is periodic. Because of this, its possible to think of the growth diagram as existing on a cylinder, but it's generally easiest to think of the diagram as extending in both directions indefinitely. All of the examples shown will use Young diagrams as the underlying poset, since it is easy to represent the order ideals as partitions, but the theory applies equally well to general posets.

One useful fact about growth diagrams is that they have a local condition which encapsulates the jeu de taquin slides and makes it easier to compute promotions in this setting.

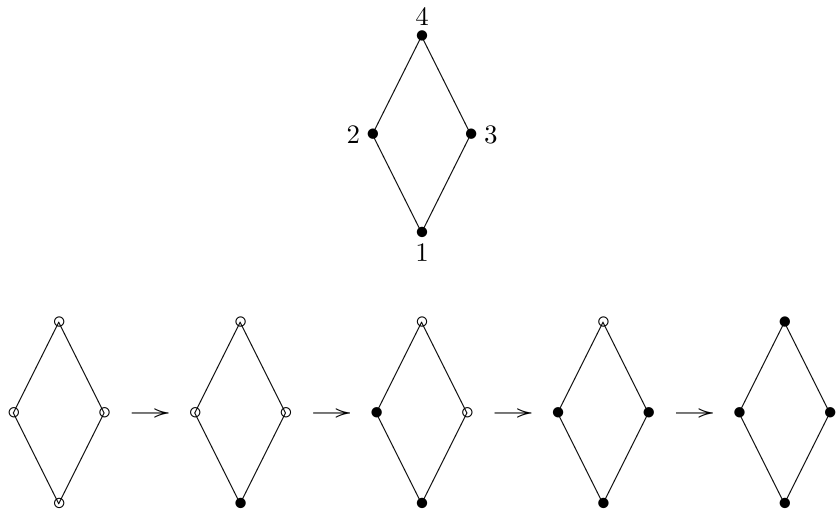
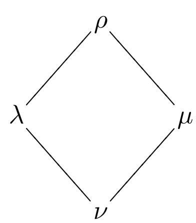


Figure 3: A linear extension of a poset, and the corresponding maximal chain the lattice of order ideals, where filled in circles represent elements of the order ideal.

Theorem 2. *If*



is a section of a growth diagram, then exactly one of the following two statements is true.

1. *If λ is the only order ideal that contains ν and is contained in ρ, then $\mu = \lambda$*

2. *There is a unique order ideal that contains ν and is contained in ρ differing from λ, and it is μ*

We will think of λ and ρ as being parts of an original maximal chain, and ν and μ as being parts of the original maximal chain's promotion. It is not immediately obvious that ν is necessarily contained in λ, or that μ is contained in ρ.. This happens because every element x in μ with some label k either has the label $k-1$ after promotion, or get's promoted and gives the label $k-1$ to some element beneath it. This means that the set of elements with labels less than k after promotion is contained in the set of elements with labels less than

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Figure 4: An example of a growth diagram for the Young diagram associated to the partition 3+2+2

$k+1$ before promotion, which is the same as saying that ν is contained in λ, or μ is contained in ρ.

Once we have this, it is not hard to verify that one of these two situations must occur. ρ is obtained from ν by adding two elements, call them x_1 and x_2. If the two elements are comparable (WLOG $x_1 < x_2$), then the only way we can get from μ to ρ would be through $\nu \cup \{x_1\}$, and then we would have $\lambda = \nu \cup \{x_1\} = \mu$. Otherwise, the two elements are incomparable, and if (WLOG) $\lambda = \nu \cup \{x_1\}$, then the unique other way to build ρ from μ is through $\mu \cup \{x_2\}$.

Now, we want to show that this local condition must be true. It suffices to consider the case where ρ, λ, and ν are known. This is because if ρ, μ, and ν are known, we have at most two options for λ. If there's only one option, then $\lambda = \mu$, and there's nothing to show. If there are two possibilities for λ, then the first case requires that λ and μ be different, so there is again only one choice for λ.

To do this, we show the element that's in an order ideal λ but not in the order ideal that's down and to the right from λ (for consistency, call it μ) represents an element that's in the chain of promotion for the maximal chain containing λ.

Lemma 3. *If λ is an element of a growth diagram, and ν is the element down and to the right from λ, then the element contained in λ not contained in ν is part of the chain of promotion for the labeling corresponding to the maximal chain containing λ.*

Proof. We will prove this lemma by induction. The base case is when ν is empty. Then λ contains a single element, which will be the element labeled 1, and by definition is in the chain of promotion of the maximal chain containing it. Now, we assume that the element contained in λ and not in ν is in the chain of promotion for the maximal chain containing λ, and we will show that the element that lies in ρ and not in μ is in the chain of promotion for the maximal chain containing ρ. To try and keep the notation tidy, we will say that λ=$\nu \cup \{x\}$, $\mu = \nu \cup \{y\}$, and $\rho = \lambda \cup \{w\} = \mu \cup \{z\}$. Our assumption is that x is in the chain of promotion for the maximal chain containing λ and ρ, and we want to show that z is also in the chain of promotion. One case is where w covers x. Say x has the label k, so w has the label $k+1$. Then since x is in the chain of promotion, and w must have the smallest label of elements covering x, then w must also be in the chain of promotion. Additionally, this means that x will have the label of k after promotion, which is to say that x will be the element we add to ν to get μ. This means we have $y = x \neq w$, so w is also the element of ρ not contained in μ. The other case would be when w doesn't cover x. This implies that w will not be part of the chain of promotion, and thus will have the label of k after promotion. w having the label of k after promotion is the same as saying $w = y$. So x will be the element of ρ not contained in μ, and by assumption x is part of the chain of promotion. $\square$

Theorem 4. Inherent in the proof of the lemma is the proof of the local condition. If there is only one way to build ρ from ν, this means that w must cover x, and in that case we saw that $y = x$, so in particular $\lambda = \mu$. If there are two ways to build ρ from ν, then x and w must be incomparable, and in that case we saw that $y = w \neq x$. Since y is not x, and there are only two order ideals between ν and ρ, this means that μ is the unique order ideal between ν and ρ that is not equal to λ. $\square$

If we're given a maximal chain arranged in an upward diagonal, we can use local condition to generate the promotion of that maximal chain by using the fact that any maximal chain in $J(P)$ starts with the empty set.

As was originally shown by Fomin, this growth diagram also generates the evacuation operator. If instead of using the local condition to all of the next maximal chain, we only use it to generate the first p elements of the next maximal chain, then the first $p-1$ element of the next adjacent maximal chain, and so on, we end up with a triangular diagram. The shortening of the maximal chains is analagous to the "freezing" of labels in the linear extension. So if we look at the diagonal going up and to the left, we see that it corresponds to the evacuation of the original maximal chain. However, the local condition is symmetric in λ and μ. Therefore, if we start with the evacuation of the original maximal chain and repeat the same process, we will get the mirror image of the previous growth diagram, which shows that evacuation is an involution. This proves part 1 of Theorem 1.

So now we have a very graphical way of intepreting the various operators in terms of this diagram, as shown in Figure 6. Say we're given a maximal chain, Q,

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 & & 2 & & 11 & & 2 & & 11 & & 2 & & 2 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset
\end{array}
$$

Figure 5: If the maximal chain on the left side of the triangle corresponds to a linear extension L, then the maximal chain on the right side of the triangle corresponds to $\epsilon(L)$.

and compute the associated growth diagram. Then promotion corresponds to a shift to the right for maximal chains written up and to the right (resp. to the left for maximal chains written up and to the left), and dual promotion corresponds to a shift to the left for maximal chains written up and to the right (resp. a shift to the right for maximal chains written up and to the left). Evacuation corresponds to the the unique maximal chain that slants the opposite direction as the original one, and meets the original one at the top of the growth diagram, and dual evacuation is the unique diagonal going the opposite direction as the original maximal chain that meets the original one at the bottom on the growth diagram.

So say we start with a maximal chain, Q, written up and to the right. Then the evacuation of this chain will be the unique diagonal going up and to the left that ends at the same place as Q, and one can easily see that the minimal element of $\epsilon(Q)$ will be p units to the right of Q. Then dual evacuation will give us a maximal chain slanted in the same direction as Q, but shifted p units to the right, which corresponds to ∂^p. Thus, we can see that $\epsilon^*\epsilon = \partial^p$, which proves part 2 of Theorem 1.

Similarly, if we start with a maximal chain Q going up and to the right, doing evacuation and then dual promotion gives the diagonal up and to the left that ends one unit to the left of where Q ends. If we instead do dual promotion and then evacuation, we see that we end up on the same diagonal, and thus proving the third part of Theorem 1.

This method of approaching promotion and evacuation seems to be a powerful tool, but it is not clear how to exploit it further beyond proving these basic relations.

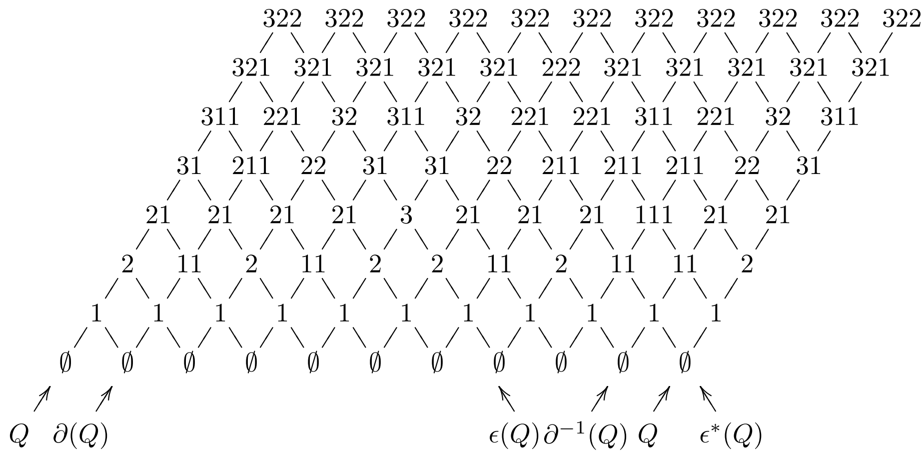


Figure 6: If the first maximal chain corresponds to a linear extension Q, then we can graphically see what $\partial(Q)$, $\partial^{-1}(Q)$, $\epsilon(Q)$, and $\epsilon^*(Q)$ are in the growth diagram.

3 Total promotion

Now, we begin to study specifically total promotion. We say that a p-element poset P is fixed by total promotion if every linear extension of P is fixed by ∂^p. We start with a basic theorem that relates the order of promotion of a poset to the order of promotion of its dual.

Theorem 4. *A p-element poset is fixed by k promotions if and only if its order dual P^* is fixed by k promotions. In particular, total promotion fixes P if and only if total promotion fixes P^*.*

Proof. If we can show that P being fixed by k promotions implies that P^* is fixed by k promotions, then we can get the reverse direction by replacing P with P^*, and using the fact that $(P^*)^* = P$. Thus, it suffices to prove the forward direction.

We will call Δ the dual operator, which takes a linear extension of P to a linear extension of P^*. If L gives an element x the label l when considered as an element of P, then $\Delta(L)$ (also refered to as L^*) will give the element x the label $p - l + 1$ when considered as an element for P^*. Clearly, this operation is an involution (i.e., $\Delta(\Delta(L)) = L$, or $(L^*)^* = L$), and also provides a bijection between $\mathcal{L}(P)$ and $\mathcal{L}(P^*)$. By looking at the definitions, one can see that $\partial_P^{-1} = \Delta \circ \partial_P^* \circ \Delta$. That is to say, dual promotion on a linear extension L of P is the same as doing dual promotion on L^* with respect to P^*, and then taking the dual linear extension. As Δ is an involution, $\partial_P^{-n} = \Delta \circ \partial_{P^*}^n \Delta$ for all n. Thus, if $\partial_P^k(L) = L$, then $L = \partial_P^k(L) = \Delta(\partial_P^k(\Delta(L))) = \Delta(\partial_P^k(L^*))$. Applying Δ to both sides of the equation yields $L^* = \partial_{P^*}^k(L)$, which says that

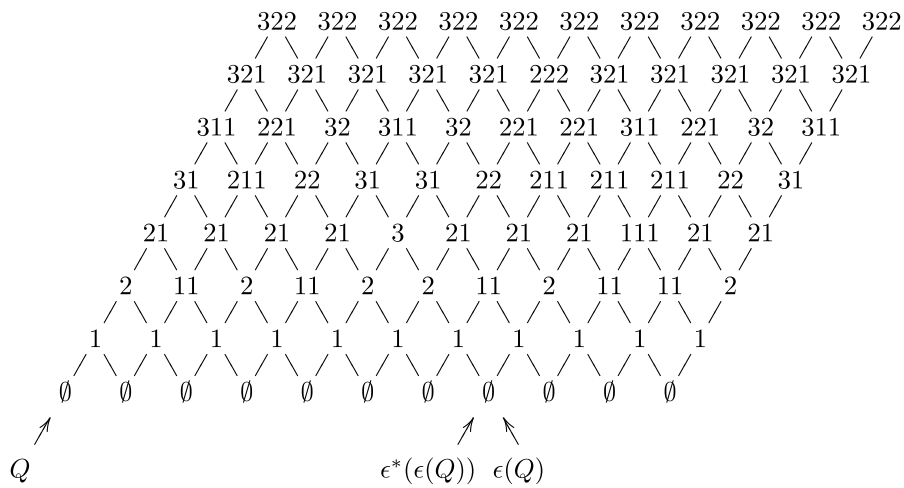


Figure 7: Graphical visualization of the relation $\epsilon^* \circ \epsilon = \partial^p$.

L^* is fixed by k promotions. Since $\partial_P^k(L) = L$ for all $L \in \mathcal{L}(P)$ by assumption, and Δ is a bijection between $\mathcal{L}(P)$ and $\mathcal{L}(P)$, we know that $\partial_{P^*}^k$ fixes all linear of P^*, as desired. □

Next, we show that the concept of being fixed by total promotion is compatible with disjoint unions.

Theorem 5. *A poset P is fixed by total promotion if and only if all of its connected components are fixed by total promotion.*

Proof. First, we'll show that if P is fixed by total promotion, then its connected components are also fixed by total promotion. Let Q be a connected component with q elements in a p-element poset P that is fixed by total promotion. We can define an induced linear extension of Q from a linear extension L of P by giving the element of Q with the smallest label under L the label 1, the element with the next smallest label under L the label 2, and so on. This linear extension will be called the *restriction of L to Q*, and will be denoted by $L|_Q$. Essentially, it keeps track of the relative order of the labels.

Fix a linear extension L' of Q. Let L be a linear extension of P such that $L|_Q = L'$. By assumption, $\partial_P^p(L) = L$. When we apply promotion to any linear extension L oif P, the chain of promotion will be entirely contained in the connected component containing the element labeled 1. So if Q doesn't contain the element labeled 1, then applying promotion to L will just decrease the labels of the elements in Q by 1, and we'll have $\partial_P(L)|_Q = L|_Q$. If Q does contain the element labeled 1, as the chain of promotion only depends on the relative order of the labelelings, the chain of promotion for L will be the same as the chain of promotion for $L|_Q$, and we'll have $\partial_P(L)|_Q = \partial_Q(L|_Q)$.

Now, consider the set of labels in the connected component Q. We notice that promotion only shifts positions of labels and then decreases them by 1, except for the label 1 which is ultimately replaced by the label p. Thus, we can see that the set of labels for a connected component are cyclically shifted modulo p by promotion. As Q contains q elements, we can see that after p cyclic shifts that the set of labels of Q will contain 1 exactly q times. Combining this with the above paragraph, we see if we do p promotion to L, we'll only change the relative order of the labels of Q q times. This means that $M' = M|_Q = \partial_P^p(M)|_Q = \partial_Q^q(M')$. Thus, the linear extension of M' of Q is fixed by total promotion.

Secondly, we prove the reverse direction. It suffices to show that the disjoint union of two connected posets that are fixed by total promotion is again fixed by total promotion. Let P be a p-element poset fixed by total promotion, and let Q be a q element poset fixed by total promotion. Consider an arbitrary linear extension of $P + Q$, L. As we observed in the first half of the proof, when we do $p + q$ promotions to the disjoint union, we only change the relative order of the labels in P p times, and similarly for Q. As P and Q are both fixed by total promotion, this means that after $p + q$ promotions to the disjoint union, the relative ordering of each connected component will be the same as it originally was. And as the set of labels for a connected component cyclically shifts modulo $p + q$, after $p + q$ promotions each connected component will have the same set of labels as it originally had. Since each connected component end ups with the same relative ordering it originally had, and the same set of labels it originally had, then the labelling we end up after $p + q$ promotions must be the same labeling we started out with. ☐

This theorem shows us that in order to study posets that are fixed by total promotion, it suffices to understand connected posets that are fixed by total promotion.

4 Enumeration of Posets Fixed Under Total Promotion

Using Maple and John Stembridge's Posets package [9], we are able to completely enumerate all posets fixed by total promotion with less than 9 elements. The following table lists how many connected posets there are for each number of vertices satisfying various properties.

The properties in the Table 2 can only be satisfied by a connected poset. The properties in the Table 1 also apply to non-connected posets, and a non-connected poset satisfies those properties if and only if all of the connected components satisfy the property.

Remark 1. By Theorem 5, we know that every poset satisfying total promotion comes from a disjoint union of connected posets that satisfy total promotion.

Number of elements	All	Ranked	Graded	Self Dual	Total number of connected posets
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	3
4	5	5	5	3	10
5	1	1	1	1	44
6	42	38	29	8	238
7	7	3	1	1	1650
8	195	155	83	23	14512
9	88	42	28	12	163341

Table 1: A count of how many connected posets fixed by total promotion satisfy various properties, along with the total number of connected posets for comparison.

So if R is a property that's compatible with disjoint union, in the sense that the statement "A poset satisfies property R if and only if all its connected components satisfy R" is true, then we can count the total number of posets fixed by total promotion satisfying such a property by looking just at connected posets. The formula for the total number of posets with n elements fixed by total promotion satisfying some property R that's compatible with disjoint union is

$$\sum_{\lambda=\lambda_1 \geq \lambda_2 \dots} \prod_i f(\lambda_i),$$

where λ is a partition of n and $f(k)$ is the number of connected posets with k elements satisfying property R.

Therefore, we can construct a similar table for all posets using the data for just the connected ones, as is done in Table 3.

5 Building Larger Posets

We have already seen that all posets fixed by total promotion can be built up from connected posets that satisfy total promotion, so now we focus our study on connected posets fixed by total promotion. We will look at what happens to total promotion when we take the ordinal sum of two posets. The ordinal sum of two posets P and Q, $P \bigoplus Q$, has elements $P \cup Q$ and $x \leq y$ in $P + Q$ if $x, y \in P$ and $x \leq y$ in P, $x, y \in Q$ and $x \leq y$ in Q, or $x \in P$ and $y \in Q$. One important property of an ordinal sum is that each minimal element of Q covers each maximal element of P. This means that for any linear extension of $P + Q$, if P has p elements and Q has q elements, then the elements that came from Q will have labels $1 \dots q$ and the elements coming from P will have labels $q + 1 \dots q + p$. When we apply the promotion operator to the ordinal sum, a

Number of elements	Bounded	Eulerian	Lattice	Distributive
1	1	0	1	1
2	1	1	1	1
3	1	0	1	1
4	2	1	2	2
5	1	0	1	1
6	10	1	9	5
7	3	0	3	1
8	44	1	28	10
9	30	0	22	2
10	206	1	119	23
11	45	0	25	1

Table 2: A count of how many posets fixed by total promotion satisfy properties which require the poset to be bounded (i.e., have a minimal and maximal element).

Number of elements	Total	Ranked	Graded	Self Dual	Total number of posets
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	5
4	9	9	9	7	16
5	11	11	10	9	63
6	60	56	42	22	318
7	74	66	46	28	2045
8	334	282	140	72	16999
9	484	378	178	102	183321

Table 3: A count of how many total (not necessarily connected) posets are fixed by total promotion and additionally satisfy each property.

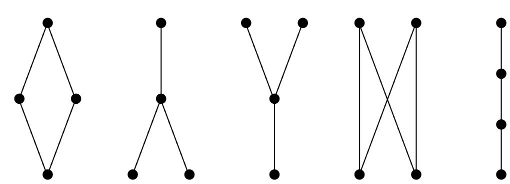
minimal element of Q will cover each of the maximal elements of P, so the next element in the chain will be the smallest labeled maximal element of P, just as if we did "sub-promotion" on P. This means that for a given linear extension of $P \cup Q$ that the chain of promotion will be the concatenation of the chain of promotion for Q with the induced linear extension and the chain of promotion for P with its induced linear extension (which can be standardized by removing q from all the labels). So each time we apply the promotion operator to $P \cup Q$, we are functionally applying the promotion operator to a copy of P and a copy of Q at the same time.

Remark 2. The ordinal sum of two posets is always connected, even if the two "summands" aren't connected. So it's possible to build up a connected poset fixed by total promotion from smaller pieces that aren't connected via ordinal sum.

This is slightly different from disjoint union, where only one of the two sub-posets would experience promotion each time we applied promotion to the union. So under disjoint union, the number of vertices scales additively, and the order of promotion likewise scales additively, which is why things work out so nicely. With an ordinal sum, the number of vertices still scales additively, but the order of promotion scales multiplicatively (specifically, it is the least common multiple of the orders of promotion of P and Q). This means we can create a poset with p elements that's fixed by total promotion by taking an ordinal sum of posets such that the order of promotion of each poset is a divisor of p, and the total number of elements in all the posets combined adds up to p.

One simple way of generating a large number of connected posets fixed by total promotion in this manner comes from compositions. A composition of a number n is a partition of n where the order of the parts matter (for example, 1+2+1 is a different composition than 2+1+1). Let $[\mathbf{n}]$ be the poset of n incomparable elements, called the *anti-chain of n elements*. All anti-chains are fixed by total promotion, as they are a disjoint union of singletons, which are trivially fixed by total promotion. Define a *strictly divisible composition of n* to be a composition of n with the property that all of its parts properly divide n, and let $\mathfrak{D}_n$ be the set of all such strictly divisible compositions.

Theorem 6. *If f is the map that takes a composition $\alpha = \alpha_1 + \alpha_2 + \dots$ to the poset $[\alpha_\mathbf{1}] + [\alpha_\mathbf{2}] + \dots$, then f takes elements of $\mathfrak{D}_n$ to posets that are fixed by total promotion.*



1+2+1 1+1+2 2+1+1 2+2 1+1+1+1

Figure 8: Strictly divisible compositions of 4 and the corresponding posets of 4 elements fixed by total promotion.

6 Minuscule Posets

One class of objects studied by Proctor are minuscule posets [4][5]. We say that ρ, a finite dimensional irreducible representation of a Lie algebra $\mathfrak{g}$ with highest weight λ, is a *minuscule representation* if every weight is of the form

$w\lambda$ for some $w \in W$. The set of weights of a minuscule representation has a standard partial order, but Proctor uses the dual of this order, where $\omega \geq \mu$ if and only if $\omega - \mu$ is a sum of positive roots. The poset derived from the weights is called an *irreducible minuscule lattice*, which Proctor proves is indeed a lattice. Finally, the poset of join-irreducible elements of an irreducible minuscule lattice is called an *irreducible minuscule poset.* As the minuscule representations of complex simple Lie algebras are classified, there is a complete classification of the irreducible minuscule posets. We will use $\mathbf{n}$ to denote the n-chain. Then the minuscule posets are $\mathbf{m} \times \mathbf{n}$, $J(\mathbf{2} \times \mathbf{n})$, $J^n(\mathbf{2} \times \mathbf{2})$, $\mathbf{2n}$, $J^2(\mathbf{2} \times \mathbf{3})$, and $J^3(\mathbf{2} \times \mathbf{3})$.

Theorem 7. *All minuscule posets are fixed by total promotion*

Proof. The first class of minuscule posets listed are rectangles, which are known to be fixed by total promotion [1][2]. The second class of minuscule posets listed are shifted shapes, which were proved to be fixed by total promotion by Haiman [2]. The third class is an $n-1$ chain ordinal summed with two incomparable elements, ordinal summed with another $n-1$. This is the poset corresponding to the composition $n = 1 + 1 + \ldots 1 + 2 + 1 + \ldots + 1$, which is fixed by total promotion by Theorem 6. The fourth class is just a chain, which is clearly fixed by total promotion (in fact, fixed by a single promotion). Finally, there are two more minuscule posets corresponding to $\mathbf{E}_6$ and $\mathbf{E}_7$, and these were verified to be fixed by total promotion using Stembridge's Posets package for Maple [9]. □

In Proctor's work, he also looks at Gaussian posets, which are defined to be posets whose rank generating function takes a special form. He proved with Stanley that all minuscule posets are Gaussian, and it is conjectured that these are the only Gaussian posets. Therefore, it would be reasonable for us to conjecture that all Gaussian posets are fixed by total promotion. It is not known if there is a direct proof that either Gaussian or minuscule posets are fixed by total promotion that use their defining properties.

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