KODAIRA DIMENSION OF LEFSCHETZ FIBRATIONS OVER TORI

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ABSTRACT. The Kodaira dimension for Lefschetz fibrations was defined in [1]. In this note we show that there exists no Lefschetz fibration over a torus with fiber genus $g \geq 3$ of Kodaira dimension 1. This proves that the Lefschetz Kodaira dimension is a diffeomorphism invariant.

1. INTRODUCTION AND DEFINITIONS

In [1] the Kodaira dimension of Lefschetz fibrations was defined and shown to be identical to the holomorphic or symplectic Kodaira dimension ($\kappa^h(M)$ resp. $\kappa^s(M)$) whenever such structures are supported. This was successful in all but one case. The goal of this note is to complete the discussion for this case.

The notion of a Lefschetz fibration is important for both symplectic and complex manifolds. In particular, it can be viewed as a topological characterization of symplectic manifolds.

Definition 1.1. *A (g, h)-Lefschetz fibration on a compact, connected, oriented smooth 4-manifold M is a map $\pi : M \to \Sigma_h$, where Σ_h is a compact, connected, oriented genus h 2-manifold and $\pi^{-1}(\partial\Sigma_h) = \partial M$, such that*

- *the set of critical points of π is isolated and lies in the interior of M;*
- *for any critical point x there are local complex coordinates (z_1, z_2) compatible with the orientations on M and Σ_h such that $\pi(z_1, z_2) = z_1^2 + z_2^2$,*
- *π is injective on the set of critical points and*
- *a regular fiber is a closed, compact, connected, oriented genus g 2-manifold.*

Furthermore, we shall assume that M is relatively minimal, i.e. there exists no fiber containing a sphere of self-intersection -1.

Lemma 1.2. [9] *For (g, h)-Lefschetz fibrations with $h \geq 1$, relative minimal is equivalent to M minimal.*

We consider only Lefschetz fibrations that have singular fibers. A singular fiber is a transversally immersed surface with a single positive double point.

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If there are no critical points, then $\pi : M \to \Sigma_h$ is just a surface bundle. We denote by k the number of singular fibers and write $k = n + s$ with s the number of separating vanishing cycles and n the number of non-separating vanishing cycles. The following relates the genus of the fiber and the base to the number of vanishing cycles.

Lemma 1.3. [2] *Let M admit the structure of a (g, h)-Lefschetz fibration with fiber genus $g \geq 2$ and base genus $h \geq 1$. Then*

$$s \leq 6(3g - 1)(h - 1) + 5n.$$

The total space M may admit symplectic structures ω: A (g, h)-Lefschetz fibration is called symplectic if there exists a symplectic form ω on M such that, for any $p \in \Sigma$, ω is non-degenerate at each smooth point of the fiber F_p and at each double point $d \in F_p$, ω is non-degenerate on the two tangent planes of the fiber F_p contained in the tangent space $T_d(M)$.

The following result shows that the Lefschetz fibrations considered in this note all carry symplectic structures and are hence almost complex.

Theorem 1.4. [3] *Suppose M is a 4-manifold admitting a (g, h)-Lefschetz fibration $\pi : M \to \Sigma_h$. If the fiber class $F \in H_2(M, \mathbb{R})$ is nontrivial, then M admits a symplectic Lefschetz fibration structure. In particular, if $g \neq 1$ then this result holds.*

The following definition of the Kodaira dimension for Lefschetz fibrations with $h \geq 1$ is purely combinatorial.

Definition 1.5. *Given a relative minimal (g, h, k)-Lefschetz fibration with $h \geq 1$, define the Kodaira dimension $\kappa^l(g, h, k)$ as follows:*

$$\kappa^l(g,h,k) = \begin{cases} -\infty & \text{if } g = 0 \ , \\ 0 & \text{if } (g,h,k) = (1,1,0) \ , \\ 1 & \text{if } (g,h,k) = (1,1,>0) \text{ or } (1,\geq 2,\geq 0) \text{ or } (\geq 2,1,0) \ , \\ 2 & \text{if } (g,h,k) = (\geq 2,\geq 2,\geq 0) \text{ or } (\geq 2,1,>0). \end{cases}$$

The Kodaira dimension of a non-minimal Lefschetz fibration with $h \geq 1$ is defined to be that of its minimal models.

Remark: Note that due to Lemma 1.2, a non-minimal Lefschetz fibration with $h \geq 1$ can only have exceptional curves in the fibers corresponding to nullhomotopic separating vanishing cycles. Hence there are at most a finite number of them and blowing these curves down preserves the fibration structure.

It was shown in [1], that $\kappa^l(g, h, k)$ coincides with the symplectic or complex Kodaira dimension on M in all cases except possibly if $(g, h, k) = (\geq 3, 1, \geq 1)$. For this remaining case M admits a symplectic structure by Theorem 1.4 and it is known that the symplectic Kodaira dimension satisfies $\kappa^s(M) \in \{1, 2\}$. To distinguish these two cases, it suffices to show that the square of the canonical class $K(M)$ (defined by the almost complex structure) satisfies either $K^2(M) = 0$ or $K^2(M) > 0$. In [1] it was conjectured that $K^2(M) = 0$ cannot happen when $g \geq 3$ and $h = 1$. This is our main result.

Lemma 1.6 (**Main Result**). *Assume that $g \geq 2$. There exists no $(g, 1, \geq 1)$-Lefschetz fibration M with $K^2(M) = 0$. Therefore, any minimal $(g, 1, \geq 1)$-Lefschetz fibration has $\kappa^l(g, 1, \geq 1) = 2$.*

In [1] it is shown that this result holds in a wide variety of cases, in particular for all hyperelliptic Lefschetz fibrations (thus covering the case $g = 2$ completely).

In the following, we will be considering Lefschetz fibrations with $h = 1$, $g \geq 3$ and $k > 0$. We shall drop k from the notation and refer only to (g, h)-Lefschetz fibrations, implicitly assuming that $k > 0$.

2. Topological Results

We begin by showing that topologically, a Lefschetz fibration over a torus of Kodaira dimension 1 behaves much like a properly elliptic fibration does.

The Euler characteristic χ of a (g, h)-Lefschetz fibration with k singular fibers is given by

$$\chi = 4(g-1)(h-1) + k \tag{1}$$

and hence, if $h = 1$ we have $\chi = k$.

On an almost complex manifold M we have the Noether formula:

$$\sigma + \chi = 0 \pmod 4$$

i.e. there exists a $\tau \in \mathbb{Z}$ such that

$$\sigma + \chi = 4\tau. \tag{2}$$

From this and (1) we obtain the following equation when $h = 1$:

$$\sigma = 4\tau - k. \tag{3}$$

Lemma 2.1. *For a $(g, 1)$-Lefschetz fibration M with $K^2(M) = 0$ we have $\tau > 0$ and thus*

$$\sigma + \chi > 0,$$
$$\sigma = -8\tau < 0,$$

and

$$\chi = k = 12\tau > 0.$$

Proof. Assuming that $K^2(M) = 0$ and recalling that M carries a symplectic structure by Theorem 1.4, we obtain from (2) that

$$0 = K^2(M) = 2\chi + 3\sigma = 8\tau + \sigma$$

from which it follows that

$$\sigma = -8\tau. \tag{4}$$

Thus

$$4\tau - k = -8\tau \;\Rightarrow\; k = 12\tau > 0. \tag{5}$$

□

An immediate corollary of Lemma 2.1 is

Corollary 2.2. *If M is a $(g,1)$-Lefschetz fibration with $\chi \neq 12\tau$ for some $\tau \in \mathbb{Z}$, then $K^2(M) > 0$. In particular, $\kappa^l(g, 1, \geq 1) = 2$.*

Remark: It is not hard to construct a $(g,1)$-Lefschetz fibration with $\chi = 12\tau$ and $K^2 > 0$. Consider the examples of $(g,0)$-Lefschetz fibrations on $N = (\Sigma_{(g-1)/2} \times S^2)\#8\overline{\mathbb{C}P^2}$ given in Section 5, [4] or Section 5, [8]. When $g = 6\tau - 5$, $\tau \geq 2$, these have $2g + 10 = 12\tau$ singular fibers. Taking the fiber sum of N and $\Sigma_g \times T^2$ gives a $(g,1)$-Lefschetz fibration M with 12τ singular fibers and $\sigma(M) = \sigma(N) = -8$, consequently $K^2(M) > 0$. Usher [10] has shown, that either M is minimal or the exceptional curves must lie in singular fibers corresponding to nullhomotopic separating vanishing cycles. There are at most finitely many such exceptional curves. After blowing down all these exceptional curves, the resulting minimal manifold has

$$K^2 \geq K^2(M) - 8 = 24(\tau - 1) > 0.$$

Thus we obtain a $(g,1)$-Lefschetz fibration with $\kappa^s(M) = 2$ and $\chi = 12\tau$. Note that this Lefschetz fibration satisfies the slope inequality of Xiao ([11]).

Assume in the following that $K^2(M) = 0$ (and thus $\chi = 12\tau$). On the base Σ_h, remove small disjoint disks around each singular point such that the fibration over each disk contains precisely one singular fiber. Thus we decompose M into a surface fibration M_p with boundary and k Lefschetz fibrations over a disk, each containing exactly one singular fiber. Thus we can decompose the signature of M by Novikov subadditivity as

$$\sigma(M) = \sigma(M_p) + \sum \sigma(\text{singular fibers}).$$

Ozbagci [6] showed that the signature of such a disk containing a separating vanishing cycle is -1, while for a non-separating vanishing cycle the contribution is 0. Hence the signature can be written as

$$\sigma(M) = \sigma(M_p) - s = -\frac{2}{3}(n + s)$$

where we have used that $K^2(M) = 0$ in the last equality. Thus

$$\sigma(M_p) = \frac{s - 2n}{3}.$$

Meyer [5] showed that the signature of a surface bundle with boundary is divisible by 4, hence

$$\sigma(M_p) = 4r$$

for some $r \in \mathbb{Z}$.

Lemma 2.3. *For a $(g,1)$-Lefschetz fibration with $K^2(M) = 0$ we have*

$$4|s \text{ and } 4|n.$$

More precisely, we have

$$s = 4r + 8\tau \text{ and } n = 4\tau - 4r$$

where τ and r are defined by $\chi = 12\tau$ and $\sigma(M_p) = 4r$.

Proof. The signature calculations above show that $s - 2n = 12r$ and hence that $12r + 3n = n + s = \chi = 12\tau$. From this it follows that

$$n = 4\tau - 4r \text{ and } s = 4r + 8\tau.$$

□

An immediate corollary is

Corollary 2.4. *Any $(g,1)$-Lefschetz fibration with $4 \nmid s$ or $4 \nmid n$ satisfies $K^2(M) > 0$. Hence any minimal such Lefschetz fibration has $\kappa^l(g,1,\geq 1) = 2$.*

Notice that this is slightly sharper than the result in Corollary 2.2.

3. Main Result

We are now ready to prove the main result.

Lemma 3.1. *There exists no $(g,1)$-Lefschetz fibration M with $K^2(M) = 0$. Therefore, any minimal $(g,1)$-Lefschetz fibration M has $\kappa^l(g,1,\geq 1) = 2$.*

Proof. Assume that M admits the structure of a $(g,1)$-Lefschetz fibration and $K^2(M) = 0$. Lemma 1.3 shows that

$$s \leq 5n$$

for this fibration. Using Lemma 2.3, it follows that

$$4r + 8\tau \leq 20\tau - 20r$$

which implies that

$$\chi = 12\tau \geq 24r = -3\sigma.$$

Hence $3\sigma + \chi \geq 0$. From Lemma 2.1 we know that $\chi > 0$, hence

$$K^2 = 3\sigma + 2\chi \geq \chi > 0$$

contradicting the assumption. Hence no such Lefschetz fibration can exist. □

Note that Lemma 1.3 is valid when $g \geq 2$, so this proof reproduces the result for $g = 2$ obtained in [1]. Clearly, for $g = 1$ this result is not true.

Note also the asymmetry in the roles of g and h in Lemma 1.3. The values of g and h appear in two calculations: The proof above and the calculation of χ. In the euler characteristic χ they appear in symmetric form, hence also for $(1, g)$-Lefschetz fibrations we obtain $\chi = k$. However, in the proof above Lemma 1.3 is used and thus this result is not valid for $(1, g)$-Lefschetz fibrations.

From these results it now follows that the Lefschetz Kodaira dimension is a diffeomorphism invariant:

Corollary 3.2. *κ^l is a diffeomorphism invariant. Moreover, when either the symplectic or the complex Kodaira dimension is defined, it coincides with the Lefschetz Kodaira dimension κ^l.*

Proof. See Proposition 4.4 in [1]. □

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